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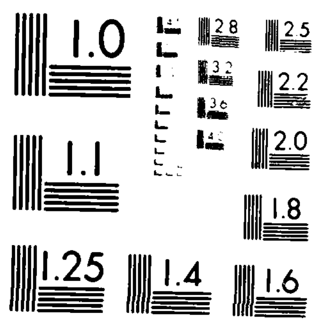

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October 1982

FUNDAMENTAL STUDIES IN SCATTERING FROM ROUGH SURFACES

Applied Science Associates, Inc.

Gary S. Brown

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
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This report presents interim results of a study to [illegible] for multiple scattering in the theoretical modeling of scattering fr[illegible] rough surfaces. Based on the magnetic field integral equation for the current induced on a perfectly conducting rough surface, a k-space integral equation for the stochastic transform of the current is derived and discussed. The mean and variance of the scattered field is shown to be directly obtainable from the stochastic transform of the current. Limiting cases of a gently

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undulating surface and a surface which is uniformly rough, i.e. a pseudo white noise surface, are considered in detail. An approximate approach is presented for determining the mean or coherent scattered field produced by a uniformly rough surface and the resulting solution clearly shows the important effects of multiple scattering. For the class of surfaces for which decorrelation does not imply statistical independence, it is found that the incoherent power scattered by such surfaces comprises two parts. The first part is the conventional diffuse term while the second part is specular in that it exists only at the specular scattering angle. For a gently undulating exponentially distributed surface, the incoherent specular power is found to exceed the coherent scattered power when the Rayleigh parameter is larger than unity. These results have significant implications with regard to the class of surface statistics to be considered for fitting measured surface statistics. Finally, the far field approximation as it applies to rough surface scattering is examined in detail for arbitrarily distributed surfaces. The failure of the approximation when the incident illumination is unbounded is explained. For a bounded incident plane wave, the far field approximation for the mean scattered field is shown to be valid provided that the illuminated area on the mean surface is very large compared to the wavelength squared. The validity of the conventional far field approximation for the variance of the scattered field requires a large illuminated area which also encompasses many decorrelation lengths on the surface. The results obtained relative to the validity of the far field approximation are general in that they hold for any degree of multiple scattering on the surface.

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1.0 INTRODUCTION

The overall objective of this research is to provide models for scattering from randomly rough terrain which can be used with a high degree of confidence in systems engineering applications. In order to accomplish this goal, two avenues of approach are actively being pursued. The first approach comprises the refinement of existing theories while the second approach entails the development of new theories which go beyond existing models.

At first glance it might appear that such a two-pronged approach does not represent an efficient use of resources; however, this is not true. For example, one may argue that existing theories should be thoroughly tested before expending efforts on advanced techniques. However, the very nature of the problem precludes a <u>rigorous</u> <u>and</u> <u>complete</u> validation of existing models because these models are based upon <u>mathematical</u> simplifications and approximation which are not fully understood. That is, a full understanding of the limitations of existing models will only come about when these models have been improved upon by overcoming some of their fundamental inadequacies. Conversely, existing models should be developed and refined to the fullest extent possible so that they can be compared to measurements for the purpose of moving toward some understanding of their <u>capabilities</u> and <u>limitations</u>.

In research of this nature where there is an immediate need for accurate models, one must be constantly alert to the pittfalls of using inherently limited models simply because they produce numbers. In fact, the technical literature on rough surface scattering is permeated with the application of models to situations which are totally outside the range of validity of the models. The apparent success of such efforts is due in large measure to the clever <u>selection</u> of the surface statistics rather than the true accuracy of the scattering model. This has led to a great deal of confusion as to which model works for what class of surfaces and, most importantly, why. Since it is extremely doubtful that the kind of surface measurements necessary as inputs to the model will ever be available, the only alternative is to turn to theoretical approaches for improving existing models. That is, for scattering situations where there is significant concern that the existing models may not be valid, it would appear that improvement of the theory is the only viable approach. For this reason, the majority of our efforts during this period have been devoted toward developing and improving upon our fundamental knowledge of rough surface scattering.

Very often when one conducts research directed toward fundamental issues it is of a very esoteric nature and it has very limited practical application. However, such is not the case with this research. In fact, after establishing some very basic techniques, it is shown that these can be used immediately to derive results which are directly applicable to real problems. For example, one often sees in the technical literature on random surface scattering use made of approximations derived from deterministic scattering theory. While such approximations must hold on a sample by sample basis, it is not clear that they pass unaltered through the averaging process necessary to construct the mean scattered field and power. With the techniques developed herein, it is possible to follow these approximations all the way through the averaging process and therefore rigorously derive their impact upon the statistical moments of the scattered field. Thus, while some of the topics addressed in this interim report may appear to be far removed from useful terrain scattering models, it must be remembered that they are the beginnings of an attempt to build a solid foundation for rough surface scattering theory which has heretofore been less than satisfactory.

Summary of Results

In Section 2.0 an integral equation for the stochastic Fourier transform of the current induced on a perfectly conducting rough surface by an incident plane wave is developed. The stochastic Fourier transform of the current is defined as the multidimensional Fourier transform with respect to all stochastic surface characteristics upon which the current depends, i.e. it is a transform from $\nabla^n\zeta, n=0,1,2,\cdots$, space to $\vec{k}_n, n=0,1,2,\cdots$, space where $\nabla^n\zeta$ represents the n^{th} order directional derivative of the surface height ζ. The integral equation is developed by multiplying the magnetic field integral equation for the current in coordinate space by a Fourier kernel involving all orders of surface height derivatives and their associated transform variables and then averaging this equation. By converting averages over $\nabla^n\zeta$ to convolutions in k_n-space, there results a singular integral equation of the first kind and of infinite dimensions. The merit of this approach centers about the fact that one can clearly see, in the integral equation, the effect of all the higher order surface height derivatives. Furthermore, it is possible to truncate the infinite dimensionality of the integral equation by retaining only those orders of surface height derivatives which have a significant variance. Relationships are also developed which show how the stochastic Fourier transform of the current may be employed to determine the statistical moments of the scattered field. Limiting cases of a gently undulating surface and a uniformly rough or pseudo white noise surface are explored relative to the average scattered field generated by them. For the uniformly rough surface, an exact one dimensional singular integral equation for the average scattered field is obtained and it is found to be very similar to the first hierarchy integral equation resulting from the diagrammatic technique. Contrary to perturbational techniques, this result does not predict that a uniformly rough or pseudo white noise surface will act as a perfect reflector.

In Section 3.0 an approximate technique is developed for estimating the average field scattered by a uniformly rough surface. The analysis is based on the remarkable similarity between the dominant terms in the magnetic field integral equation for the current for both the uniformly rough and the gently undulating surfaces, i.e. the only essential difference relative to the mean scattered field is the effect of the grossly different surface correlation lengths for the two cases. This similarity suggests that if the small correlation length associated with the uniformly rough surface could be mathematically introduced into the description of the scattering associated with the gently undulating surface and if the latter scattering problem could be solved then it should be a good approximation to the uniformly rough surface scattering process. The essence of the approach is mathematical similitude; that is, the technique in itself does not necessarily have any physical interpretation. Consequently, it is absolutely essential that the results are capable of being put into one to one correspondence with one's physical understanding of the scattering process. To accomplish the introduction of the artificial correlation length into the scattering description for the gently undulating surface, the gently undulating surface is replaced by a discrete approximation comprising large, non-overlapping, flat areas having random elevations with respect to the mean planar surface. The average scattered field produced by this approximate surface is found to depend upon the degree of correlation between adjacent large flat areas. It is at this point in the development that the artificial correlation length is introduced, i.e. rather than taking the large areas to be highly correlated corresponding to the true gently undulating surface they are, instead, taken to be uncorrelated. The resulting average scattered field is subsequently found to depend upon the number of interacting areas and their height variance. Furthermore, the results show that the number of interacting areas which give rise to a maximum scattered field increases with surface roughness. Physically, this observation corresponds to the fact that as the roughness increases so does t e degree of multiple scattering on the surface, i.e. an increase in multiple scattering implies an increase in the number of effective interacting areas on the discrete artificial or substitute surface. Thus, the results of the analysis do indeed appear to have a very good physical basis. Comparison of numerical results with approximate results from the diagrammatic approach show very good agreement over the range of Rayleigh parameter where one should expect agreement.

In Section 4.0 the problem of scattering from surfaces which are neither Gaussian distributed nor do they become statistically independent as they become decorrelated is considered. This is a very important practical problem because there is certainly no guarantee that real terrain is adequately represented by a surface roughness which is Gaussian distributed. The stochastic Fourier transform of the current, developed in Section 2.0, is used to derive exact expressions for the mean and variance of the scattered field. For surfaces in which decorrelation does not imply statistical independence it is found that the variance of the scattered field or the incoherent power has two distinct parts. The first part is the conventional so-called diffuse power which is related to the spatial Fourier transform of the two point joint characteristic function for the random surface characteristics. The second part of the incoherent power is specular in its angular behavior and it is determined by the difference between the two point joint characteristic functions for decorrelation and statistical independence. A numerical comparison of this specular incoherent power term with the mean scattered field for a gently undulating exponentially distributed surface is also presented. When the Rayleigh parameter is much smaller than unity, the power associated with the mean scattered field is the larger quantity. Near a Rayleigh parameter value of one, the two quantities are nearly equal. When the Rayleigh parameter greatly exceeds one, the incoherent specular power becomes much larger than the power associated with the mean scattered field. The importance of these results to the conventional interpretation of scattering measurements are discussed in depth. These results also indicate that extreme caution should be used in fitting measured surface statistics to functional forms for the probability density function in which decorrelation does not imply statistical independence.

Section 5.0 considers the applicability of the far field approximation to scattering from infinite rough surfaces when the incident plane wave illumination is either bounded or unbounded. This also is a very important theoretical and practical problem because rough surface scattering measurements are always interpreted using this approximation and yet its validity has only been partially demonstrated for very special surfaces. The analysis once again uses the stochastic Fourier transform of the current in conjunction with the exact expression for the scattered field. For the case of an incident plane wave which is unbounded, it is shown that the mean or average scattered field is a redirected plane wave propagating in the specular direction. The amplitude of this plane wave is attenuated relative to the incident field by the effects of the surface roughness. Furthermore, its polarization is dependent upon the multiple scattering processes occurring on the surface. The variance of the scattered field or the incoherent power in any one direction is found to depend upon a <u>weighted average</u> of the angular spectrum over all angles (both visible and invisible) or directions. This result clearly shows that the concept of a far field is <u>not applicable</u> to the case of unbounded illumination because in the far field approximation the scattered power should be determined by one and only one Fourier component of the angular spectrum. When the incident illumination is bounded, the far field approximation for the mean scattered field is found to be valid provided the cross sectional area of the incident field is very large compared to the square of the electromagnetic wavelength. This caveat is necessary to insure that the difference between the support of the incident illumination and the support of the current is not significant. These two supports will be different when multiple scattering on the surface is important. For the variance of the scattered field, it is further necessary to assume that the support of the incident illuminaiton encompasses many surface decorrelation intervals in order to bring the exact result into agreement with the far field approximation. Finally, the exact results are used to show why the far field approximation breaks down when the incident illumination is unbounded.

All of the following material has either been accepted for publication in journals or it is presently in the review process for publication in a journal. Consequently, the style of the sections is tailored to the demands of the particular journal. In order to avoid a complete retype of the material, it was decided to include it in this report as it was sent to a journal. This choice, hopefully, will not be too confusing to the reader.

A Stochastic Fourier Transform Approach To Scattering

From Perfectly Conducting Randomly Rough Surfaces

by

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Abstract

An exact alternative approach to the diagrammatic technique for treating scattering from rough surfaces is developed. The magnetic field integral equation for the current induced on the rough perfectly conducting surface is multiplied by a Fourier kernel involving all orders of surface height derivatives and their associated transform variables. Averages of this weighted equation are converted to convolutions in the transform domain. The result of this operation is a singular integral equation of the first kind of infinite dimensions (because of the infinite number of height derivatives) for the stochastic Fourier transform of the current. A procedure is developed for estimating the effects of ignoring one or more surface height derivatives in terms of the range of validity of the resulting approximate solution. Special limiting cases of very gently undulating surfaces and uniformly rough surfaces are examined. New and illuminating results are obtained for the latter case.

Introduction and Summary

There is a very definite need to move beyond the approximate rough surface scattering solutions provided by physical optics and boundary perturbation theory. The first clear indication of this need was the failure of physical optics and perturbation theory to accurately predict the magnitude of the average field [1] and acoustic pressure [2] scattered by a roughened water surface for large surface roughness. Measurements of the incoherent average scattered power [3] have also provided results which are not in agreement with an appropriate combination of the above noted single scattering approximations. Of course it is one thing to recognize the need for a less approximate solution but quite another to obtain such a solution!

Recently, a significant advance toward this goal has been provided by the work of Zipfel and DeSanto [4] and subsequent work by DeSanto [5,6]. The results obtained in these analyses were based upon two very important points which should be emphasized. First, by recasting the coordinate space integral equation <u>in the transform domain</u>, the authors were able to circumvent problems associated with the stochastic nature of the coordinate space integral equation for the "surface" Green's function. Second, by using cluster decomposition and partial summation techniques it was possible to express the solution of the transform space integral equation in terms of integral equations involving successively higher orders of surface <u>and</u> field interaction. Numerical solution of the integral equation involving the first order interaction for the average scattered field (the zeroth order interaction being the physical optics approximation) has clearly shown the physical optics approximation to be inadequate for large surface roughness and small correlation length [7]. The strength of this result lay primarily in the rigorous nature of the approach and, consequently, the generality of the solution. Unfortunately, the solution does not lend itself to an obvious physical interpretation. Furthermore, it is also not always easy to see the direct effect of the surface's statistical characteristics in the higher order integral equations.

The purpose of this paper is to present an approach and a solution which provides a bit more physical insight into certain aspects of the problem. In particular, one of the strong points of this approach is that it allows a direct correlation between surface statistical approximations and the range of validity of the resulting solution. This is an important result because it allows one to estimate when an approximate solution is valid and when it breaks down based upon the statistical characteristics of the rough surface.

The analysis is based upon the magnetic field integral equation (MFIE) for the current induced on the surface of a perfectly conducting body. Since any statistical moment of the scattered field can be obtained from knowledge of the current, this is a logical starting point. A key element in this or any analysis involving stochastic integral equations is the technique employed to overcome the stochastic nature of the integral equation. The approach used here is to multiply the MFIE by the Fourier kernel $\exp\{jk_1\zeta + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta\}$ (where $\zeta(x,y)$ is the stochastic surface height, $\nabla\zeta(x,y)$ are the stochastic surface slopes, etc.) and then average the resulting equation over ζ, $\nabla\zeta$, etc. The averages, however, are converted to convolutions of the terms in the integral equation and the Fourier transform of the appropriate joint probability density function for the stochastic variables ζ, $\nabla\zeta$, etc. Thus, whereas Zipfel and DeSanto [4] used a coordinate space transform to overcome the difficulties associated with the stochastic nature of the problem, this approach transforms all of the stochastic variables (ζ, $\nabla\zeta$, etc.) into a non-stochastic space (k_1, $\vec{k}_2$, etc.). The formal result of these and other manipulations are coupled integral equations for the x and y components of the current which are Fredholm equations of the first kind. The dimensionality of these two integral equations is formally infinite because the current depends, in general, upon ζ, $\nabla\zeta$, etc. The explicit dependence of the integral equations upon the surface statistics is contained in the single and two-point joint characteristic functions for the variables ζ, $\nabla\zeta$, etc. By examining the way in which the surface statistics ($\langle\zeta^2\rangle$, $\langle(\nabla\zeta)^2\rangle$, etc.) appear in the characteristic functions, it is possible to translate the neglect of higher order surface height derivatives directly into criteria for the range of validity of the resulting approximate solution. An orderly procedure for relating mathematical simplifications to the surface statistics is presented.

As an example of the power of this technique, it is applied to surfaces having a correlation length much larger than the electromagnetic wavelength (very gently undulating surfaces) and to surfaces having a correlation length much smaller than a wavelength (uniformly rough surfaces). For very gently undulating surfaces, the technique provides results which are in agreement with the scalar single scattering or physical optics approximation. For uniformly rough surfaces, the stochastic Fourier transform approach is found to yield two parts to the slope normalized surface current, i.e. $\vec{J}_s(\vec{r})\sqrt{1+|\nabla\zeta|^2}$ where $\vec{J}_s$ is the current. One part is found to be a function of only the stochastic surface height while the other part depends on the surface height and slopes but in such a manner as to contribute nothing to the average field scattered by the surface. An integral equation for the average scattered field is derived for the uniformly rough surface and it is found to be similar to the approximate equation obtained previously by Zipfel and DeSanto [4]. Finally, it is argued that the gently undulating and uniformly rough cases provide lower and upper bounds, respectively, on the average scattered field for a fixed height variance.

Approach Rationale

The stochastic surface height $\zeta(x,y)$ is defined with respect to the $z = 0$ plane and is taken to be a zero mean statistically homogeneous process. The region above the surface is free-space while the medium beneath the interface is perfectly conducting. For a "sufficiently well behaved" surface, the current $\vec{J}_s$ induced on the surface by an incident magnetic field $\vec{H}^i$ is given by the so-called magnetic field integral equation [8, pg. 354];

$$\vec{J}_s(\vec{r}) = 2\hat{n}(\vec{r}) \times \vec{H}^i(\vec{r}) + \frac{1}{2\pi}\int_{S_o} \hat{n}(\vec{r}) \times \left[\vec{J}_s(\vec{r}_o) \times \nabla_o g(|\vec{r}-\vec{r}_o|)\right] dS_o \qquad (1)$$

where $\hat{n}(\vec{r})$ is the upward directed unit normal to the surface and $g(|\vec{r}-\vec{r}_o|)$ is proportional to the free space scalar Green's function, i.e.

$$\hat{n}(\vec{r}) = \left[-\zeta_x\hat{x} - \zeta_y\hat{y} + \hat{z}\right] \Big/ \sqrt{1 + \zeta_x^2 + \zeta_y^2}$$

$$g(|\vec{r} - \vec{r}_o|) = \exp(-j\, k_o|\vec{r} - \vec{r}_o|) \Big/ |\vec{r} - \vec{r}_o|$$

and $k_o = 2\pi/\lambda_o$ is the free-space wavenumber. The quantities $\zeta_x = \partial\zeta/\partial x$ and $\zeta_y = \partial\zeta/\partial y$ are the x and y components of the surface slope. Expanding the double cross product in (1), converting the surface integration to an integration over the $z_o = 0$ plane through $dS_o = \sqrt{1 + \zeta_{x_o}^2 + \zeta_{y_o}^2}\, d\vec{r}_{t_o}$, and multiplying both sides of (1) by $\sqrt{1 + \zeta_x^2 + \zeta_y^2}$ yields the following;

$$\vec{J}(\vec{r}) = 2\vec{N}(\vec{r}) \times \vec{H}^i(\vec{r}) + \frac{1}{2\pi} \int \left\{[\vec{N}(\vec{r})\cdot\nabla_o g]\vec{J}(\vec{r}_o) - [\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]\nabla_o g\right\} d\vec{r}_{t_o} \qquad (2)$$

where

$$\vec{J}(\vec{r}) = \sqrt{1 + \zeta_x^2 + \zeta_y^2}\; \vec{J}_s(\vec{r})$$

$$N(\vec{r}) = \sqrt{1 + \zeta_x^2 + \zeta_y^2}\; \hat{n}(\vec{r})$$

$$d\vec{r}_{t_o} = dx_o\, dy_o$$

and the integration is over the entire $z_o = 0$ plane. Since the current must be tangential to the surface and $\sqrt{1 + \zeta_x^2 + \zeta_y^2} > 0$, $\vec{N}(\vec{r}_o)\cdot\vec{J}(\vec{r}_o) = 0$ and the z-component of the current may be expressed in terms of the x and y components as follows;

$$J_z(\vec{r}_o) = \zeta_{x_o} J_x(\vec{r}_o) + \zeta_{y_o} J_y(\vec{r}_o) \qquad (3)$$

Substituting (3) in the right side of (2) yields two coupled integral equations for $J_x(\vec{r})$ and $J_y(\vec{r})$. The coupling is a consequence of the term $[\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]\nabla_o g$

which, using (3) to eliminate $J_z(\vec{r}_o)$, has the following x and y-components;

$$[\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]\,\frac{\partial g}{\partial q_o} = -\left\{(\zeta_x - \zeta_{x_o})J_x(\vec{r}_o) + (\zeta_y - \zeta_{y_o})J_y(\vec{r}_o)\right\}\frac{\partial g}{\partial q_o} \tag{4}$$

where q_o is x_o or y_o . Thus, it is the x and y-components of (2) along with (3) which provide the essential relationships for the vector components of the current.

Having found the current, it is then possible to determine the far-zone scattered electric field in the direction $\hat{k}_s$ and at a distance R_o from the $z = 0$ plane via the following relationship [8,pg. 361]

$$\vec{E}_s = j\,\frac{k_o\,\eta_o}{4\pi}\,g(R_o)\hat{k}_s \times \hat{k}_s \times \int \vec{J}(\vec{r})\exp(j\,\vec{k}_s\cdot\vec{r})\,d\vec{r}_t \tag{5}$$

where $\vec{r} = x\hat{x} + y\hat{y} + \zeta(x,y)\hat{z}$, $\vec{k}_s = k_{sx}\hat{x} + k_{sy}\hat{y} + k_{sz}\hat{z}$, $d\vec{r}_t = dxdy$, and the integration is over the $z = 0$ plane. Of particular importance however are the statistical moments of the scattered field such as $\langle\vec{E}_s\rangle$ and $\langle|\vec{E}_s|^2\rangle$ where $\langle\cdot\rangle$ denotes an average over <u>all</u> random quantities. In view of the factor $\exp(jk_{sz}\zeta)$ in (5), it would do no good to compute $\langle\vec{J}(\vec{r})\rangle$ from (2) and (3) because the average of the product is not, in general, equal to the product of the averages. One could multiply (2) and (3) by $\exp(j\,k_{sz}\zeta)$ and average the resulting equations but the result would only be useful in computing $\langle\vec{E}_s\rangle$. That is, higher order field moments would require a new solution of some appropriately tailored form of (2) and (3). This type of reasoning leads to the conclusion that what is needed is a one-time solution of (2) and (3) which could be used in (5) or any order self-product of (5).

The average of (5) entails multiplying by the probability density function (pdf) for <u>all</u> stochastic variables upon which $\vec{J}$ depends and then integrating the product over all stochastic variables, e.g.

$$\langle\vec{J}(\vec{r})\exp(j\,k_{sz}\,\zeta)\rangle = \int_{-\infty}^{\infty}\!\!\cdots\!\int \vec{J}(\vec{r}_t,\zeta,\nabla\zeta,\ldots)\exp(j\,k_{sz}\,\zeta)p_1(\zeta,\nabla\zeta,\ldots)$$

$$\cdot\ d\zeta\ d\nabla\zeta\cdots \tag{6}$$

where it has been assumed that the order of integration [over r_t in (5)] and averaging can be arbitrarily interchanged. The surface slopes ($\nabla\zeta$), curvatures ($\nabla^2\zeta$), rate of change of curvatures ($\nabla^3\zeta$), etc. are included because, in general, the current is a function of these stochastic variables. Starting first with the integral over ζ in (6), it is obvious that this is identical to the Fourier transform (with respect to ζ) of the product of the current and the pdf. Since the Fourier transform of the product of two functions is the convolution of the transforms, the ζ-integration can be expressed as follows;

$$\int_{-\infty}^{\infty}\vec{J}(\vec{r}_t,\zeta,\nabla\zeta,\ldots)p_1(\zeta,\nabla\zeta,\ldots)\exp(j\,k_{sz}\,\zeta)\,d\zeta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\vec{J}}(\vec{r}_t,k_1,\nabla\zeta,\nabla^2\zeta,\ldots)\tilde{p}_1(k_{sz}-k_1,\nabla\zeta,\nabla^2\zeta,\ldots)dk_1 \tag{7}$$

where the tilde denotes the Fourier transform with respect to ζ. In a similar manner, the integrations over the slopes($\nabla\zeta$), curvatures($\nabla^2\zeta$), etc. can also be written as convolutions so that (6) becomes (with $s_n = \sum_{i=1}^{n} i$)

$$\langle\vec{J}(\vec{r})\exp(j\,k_{sz}\,\zeta)\rangle = \lim_{n\to\infty}\frac{1}{(2\pi)^{s_n}}\int\cdots\int\tilde{\vec{J}}(\vec{r}_t,k_1,\vec{k}_2,\ldots,\vec{k}_n)$$

$$\cdot\ \tilde{p}_1(k_{sz}-k_1,-\vec{k}_2,\ldots,-\vec{k}_n)dk_1\,d\vec{k}_2\ldots d\vec{k}_n \tag{8}$$

where $\vec{k}_n$ is an n-dimensional transform vector associated with $\nabla^n\zeta$ and the tilde implies Fourier transforms with respect to ζ, $\nabla\zeta$, $\nabla^2\zeta$, etc. If the second moment of the scattered field, $<|\vec{E}_s|^2>$, is desired, it may be obtained from knowledge of the following products;

$$<J_q(\vec{r})J_q^*(\vec{r}')\exp[jk_{sz}(\zeta-\zeta')]> = \lim_{n\to\infty} \frac{1}{(2\pi)^{s_n^2}} \int\cdots\int \tilde{J}_q(\vec{r}_t,k_1,\vec{k}_2,\ldots,\vec{k}_n)$$

$$\cdot\left\{\tilde{J}_q(\vec{r}'_t,-k'_1,-\vec{k}'_2,\cdots,-\vec{k}'_n)\right\}^*\tilde{\tilde{p}}_2(k_{sz}-k_1,-k_{sz}-k'_1,-\vec{k}_2,-\vec{k}'_2,\cdots,-\vec{k}_n,-\vec{k}'_n;\vec{r}_t-\vec{r}'_t)dk_1dk'_1\cdots d\vec{k}_nd\vec{k}'_n \quad (9)$$

where * denotes the complex conjugate and $p_2(\cdot)$ is the two-point joint probability density function. In obtaining the above result, use has been made of the fact that $F[J_q^*(\zeta,\nabla\zeta,\cdots)] = \left\{\tilde{J}_q(-k_1,-\vec{k}_2,\cdots)\right\}^*$ where F denotes the Fourier transform operation. The Fourier transforms of the single and two point joint probability density functions are the single and two point joint characteristic functions, respectively [9, pg. 254].

It is obvious from (8) and (9) that if the Fourier transform of the current with respect to the stochastic variables can be found then $<\vec{E}_s>$ and $<|\vec{E}_s|^2>$ may be determined by a straightforward integration of known functions. Thus, the problem reduces to solving for the stochastic Fourier transform of the current using (2) and (3). Unfortunately, one cannot simply take the stochastic Fourier transform of both sides of (2) in a direct manner; such an operation must be accomplished within the framework of probability theory. That is, both sides of (2) are first multiplied by the stochastic Fourier kernel $\exp\{jk_1\zeta + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta\}$ and then averaged over all stochastic variables. The averaging operation weights each term in (2) by an appropriate probability density function. For example, the left side of (2) and the source term on the right side are weighted by the single point joint pdf while the integral term is multiplied by the appropriate two-point joint pdf. The averages over $\zeta,\nabla\zeta,\cdots$ and $\zeta_o,\nabla_o\zeta_o,\cdots$ are then converted to integrations with respect to the transform variables by means of convolution identities. The net result is an infinite dimensional Fredholm integral equation of the first kind for the stochastic Fourier transform of the current. This is the essence of the stochastic Fourier transform technique.

Detailed Development

While the basic concept of the stochastic Fourier transform approach is straightforward, there are a number of details associated with converting (2) into the appropriate integral equation which require amplification. Furthermore, there is at least one intermediate result which has significant implications and should, therefore, be derived in an orderly manner. Since the conversion of (2) requires dealing with an infinite number of stochastic variables $(\zeta, \nabla\zeta, \nabla^2\zeta, \cdots)$ and their associated transform coordinates $(k_1, \vec{k}_2, \vec{k}_3, \cdots)$, the algebra and symbolish is very tedious especially because of the term $[\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]\nabla_o g$. Ignoring this term in (2) does not impact the generality of the approach because if one understands how the term $[\vec{N}(\vec{r})\cdot\nabla_o g]\vec{J}(\vec{r})$ is treated then inclusion of the neglected term can be accomplished almost by inspection. Thus, ignoring the coupling term in (2) yields the following two decoupled scalar equations;

$$J_q(\vec{r}) = 2\hat{q}\cdot[\vec{N}(\vec{r}) \times \vec{H}^i(\vec{r})] + \frac{1}{2\pi}\int\left\{-\zeta_x\frac{\partial g}{\partial x_o} - \zeta_y\frac{\partial g}{\partial y_o} + \frac{\partial g}{\partial \zeta_o}\right\}J_q(\vec{r}_o)d\vec{r}_{t_o} \quad (10)$$

where q = x or y. From a practical point of view, ignoring the coupling term in (2) implies that the curvatures $(\nabla^2\zeta)$ and higher order derivatives of the surface height, according to (4), are negligibly small; however, this simplification will not be used in this section.

Multiplying the left side of (10) by the product of the Fourier kernel

$\exp\{jk_1\zeta + j \sum_{n=1}^{\infty} \vec{k}_{n+1}\cdot\nabla^n\zeta\}$ and the single point probability density $p_1(\zeta,\nabla\zeta,\nabla^2\zeta,\cdots)$, integrating over all the stochastic variables to form the average, and converting these integrations into convolutions in the Fourier transform domain yields

$$\langle J_q(\vec{r}) \exp[jk_1\zeta + \sum_{n=1}^{\infty} \vec{k}_{n+1}\cdot\nabla^n\zeta]\rangle = \lim_{n\to\infty} \frac{1}{(2\pi)^{s_n}} \int\cdots\int \tilde{J}_q(\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots\vec{\beta}_n)$$

$$\cdot\ \tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n)\, d\beta_1 d\vec{\beta}_2\cdots d\vec{\beta}_n \tag{11}$$

where $\vec{r}_t = x\hat{x} + y\hat{y}$, $\tilde{J}_q(\cdot)$ is the stochastic Fourier transform of $J_q(\vec{r})$, and $\tilde{p}_1(\cdot)$ is the characteristic function for the single point joint pdf, i.e. the stochastic Fourier transform of the joint pdf. Taking the incident magnetic field on the surface to have the form

$$\vec{H}^i(\vec{r}) = H_o\,\hat{h}\exp(-j\,\vec{k}_{i_t}\cdot\vec{r}_t - j\,k_{iz}\,\zeta) \ , \tag{12}$$

the term $2\hat{q}\cdot[\vec{N}(\vec{r})\times\vec{H}^i(\vec{r})]$ may be expressed as follows;

$$2\hat{q}\cdot[\vec{N}(\vec{r})\times\vec{H}^i(\vec{r})] = 2H_o[C_z^q + C_x^q\zeta_x + C_y^q\zeta_y]\exp(-j\,\vec{k}_{i_t}\cdot\vec{r}_t - j\,k_{iz}\,\zeta) \tag{13}$$

where $\vec{k}_i = \vec{k}_{i_t} + k_{iz}\hat{z}$ is the incident wave vector of magnitude k_o and direction $\hat{k}_i$. The factors C_z^q, C_x^q, and C_y^q are determined by the polarization of the incident magnetic field, i.e. $C_z^q = \hat{q}\cdot(\hat{z}\times\hat{h})$, $C_x^q = -\hat{q}\cdot(\hat{x}\times\hat{h})$, $C_y^q = -\hat{q}\cdot(\hat{y}\times\hat{h})$, and are independent of the spatial $(\vec{r}_t)$ and stochastic $(\zeta,\nabla\zeta,\cdots)$ variables. The average of the product of (13) and the Fourier kernel is a straightforward multidimensional Fourier transform, i.e.

$$\langle 2\hat{q}\cdot[\vec{N}(\vec{r}) \times \vec{H}^{1}(\vec{r})] \exp[jk_1\zeta + j\sum_{n=1}^{\infty} \vec{k}_{n+1}\cdot\nabla^n\zeta]\rangle$$

$$= 2H_o \exp(-j\vec{k}_{1_t}\cdot\vec{r}_t)\left[C_z^q + jC_x^q\partial_{k_{2x}} + jC_y^q\partial_{k_{2y}}\right]$$

$$\cdot\ \tilde{p}_1(k_1 - k_{1z}, \vec{k}_2, \vec{k}_3, \cdots) \tag{14}$$

where $\vec{k}_2 = k_{2x}\hat{k}_{2x} + k_{2y}\hat{k}_{2y}$. The derivatives with respect to k_{2x} and k_{2y} are a consequence of the slope terms ζ_x and ζ_y , respectively, in (13).

Averaging the integral term in (10) requires the general two-point probability density function $p_2(\zeta,\zeta_o,\nabla\zeta,\nabla_o\zeta_o,\nabla^2\zeta,\nabla_o^2\zeta_o,\cdots;\vec{r}_t - \vec{r}_{t_o})$ because multiplication by the Fourier kernel introduces the additional stochastic variables $\nabla^2\zeta,\nabla^3\zeta,\cdots$ which are not contained in the kernel of the integral equation (10). Thus, the average of the product of the Fourier kernel and the integral term in (10) may be written as follows;

$$\left\langle \int\left\{-\zeta_x\frac{\partial g}{\partial x_o} - \zeta_y\frac{\partial g}{\partial y_o} + \frac{\partial g}{\partial\zeta_o}\right\} J_q(\vec{r}_{t_o},\zeta_o,\nabla_o\zeta_o,\cdots)\exp\left[jk_1\zeta_1 + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta\right]d\vec{r}_{t_o}\right\rangle$$

$$= \int\cdots\int\left\{-\zeta_x\frac{\partial g}{\partial x_o} - \zeta_y\frac{\partial g}{\partial y_o} + \frac{\partial g}{\partial\zeta_o}\right\} J_q(\vec{r}_{t_o},\zeta_o,\nabla_o\zeta_o,\cdots)$$

$$\cdot\ \exp\left[jk_1\zeta_1 + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta\right] p_2(\zeta,\zeta_o,\nabla\zeta,\nabla_o\zeta_o,\cdots)\,d\vec{r}_{t_o}\,d\zeta\,d\zeta_o\,d\nabla\zeta\,d\nabla_o\zeta_o\cdots \tag{15}$$

Assuming that the orders of integration may be arbitrarily interchanged, the ζ-integration can be written as a convolution of the ζ-Fourier transforms of the Green's function derivatives and $p_2(\cdot)$. Noting that

$$F_\zeta\left\{\frac{\partial g(\vec{r}_t - \vec{r}_{t_o},\zeta-\zeta_o)}{\partial\zeta_o}\right\} = -\exp(j\beta_o\zeta_o)\,F_\zeta\left\{\frac{\partial g(\vec{r}_t - \vec{r}_{t_o},\zeta)}{\partial\zeta}\right\}$$

where F_ζ denotes the Fourier transform with respect to ζ and β_o is the transform variable, and substituting

$$\tilde{g}(\Delta\vec{r}_t,\beta_o) = F_\zeta\{g(\Delta\vec{r}_t,\zeta)\} = \int g(\Delta\vec{r}_t,\zeta)\exp(j\beta_o\zeta)\,d\zeta$$

$$\tilde{g}_\zeta(\Delta\vec{r}_t,\beta_o) = F_\zeta\left\{\frac{\partial g(\Delta\vec{r}_t,\zeta)}{\partial\zeta}\right\} = \int\frac{\partial g(\Delta\vec{r}_t,\zeta)}{\partial\zeta}\exp(j\beta_o\zeta)\,d\zeta$$

in the convolution integrations with $\Delta\vec{r}_t = \vec{r}_t - \vec{r}_{t_o}$ reduces (15) to the following form;

$$\langle\cdot\rangle = \frac{1}{2\pi}\int\cdots\int\left\{-\zeta_x\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial x_o} - \zeta_y\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial y_o} - \tilde{g}_\zeta(\Delta\vec{r}_t,\beta_o)\right\}$$

$$\cdot\, J_q(\vec{r}_{t_o},\zeta_o,\nabla_o\zeta_o,\cdots)\exp\left[j\beta_o\zeta_o + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta\right]$$

$$\cdot\,\tilde{p}_2(k_1-\beta_o,\zeta_o,\nabla\zeta,\nabla_o\zeta_o,\cdots;\Delta\vec{r}_t)\,d\beta_o\,d\vec{r}_{t_o}\,d\zeta_o\,d\nabla\zeta\,d\nabla_o\zeta_o\cdots \qquad (16)$$

where the tilde symbol denotes the Fourier transform of $p_2(\cdot)$ with respect to ζ .

The ζ_o-integrations in (16) may be expressed as convolutions with a shifted argument due to the exponential factor in (16). The $\nabla_o\zeta_o$, $\nabla_o^2\zeta_o,\cdots$ integrations may be represented by convolutions with no shift in the argument. Finally, the integrations over $\nabla\zeta,\nabla^2\zeta,\cdots$ may be written as straightforward Fourier transforms. Accomplishing all of these operations yields the following result;

$$\langle\cdot\rangle = \lim_{n\to\infty} \frac{1}{(2\pi)^{2+s_n}} \int\cdots\int \left\{ j \frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial x_o} \partial_{k_{2x}} + j \frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial y_o} \partial_{k_{2y}} - \tilde{g}_{\zeta}(\Delta\vec{r}_t,\beta_o) \right\}$$

$$\cdot\ \tilde{J}_q(\vec{r}_{t_o},\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,\vec{k}_n,-\vec{\beta}_n;\Delta\vec{r}_t)d\vec{r}_{t_o}\,d\beta_o d\vec{\beta}_1\cdots d\vec{\beta}_n \tag{17}$$

where

$$\tilde{J}_q(\vec{r}_{t_o},\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n) = \int\cdots\int J_q(\vec{r}_{t_o},\zeta_o,\nabla_o\zeta_o,\cdots,\nabla_o^n\zeta_o)\exp\left\{j\beta_1\zeta_o + j\sum_{i=1}^{n}\vec{\beta}_{n+1}\cdot\nabla_o^n\zeta_o\right\}$$

$$\cdot\ d\zeta_o\, d\nabla_o\zeta_o\cdots d\nabla_o^n\zeta_o \tag{18}$$

and

$$\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,\vec{k}_n,-\vec{\beta}_n;\Delta\vec{r}_t) = \int\cdots\int p_2(\zeta,\zeta_o,\nabla\zeta,\nabla_o\zeta_o,\cdots,\nabla^n\zeta,\nabla_o^n\zeta_o;\Delta\vec{r}_t)$$

$$\cdot\ \exp\left\{j(k_1-\beta_o)\zeta + j(\beta_o-\beta_1)\zeta_o + j\sum_{i=1}^{n}\vec{k}_{n+1}\cdot\nabla^n\zeta - j\sum_{i=1}^{n}\vec{\beta}_{n+1}\cdot\nabla_o^n\zeta_o\right\}$$

$$\cdot\ d\zeta\, d\zeta_o\, d\nabla\zeta\, d\nabla_o\zeta_o\cdots d\nabla^n\zeta\, d\nabla_o^n\zeta_o \tag{19}$$

The average of (10) weighted by the Fourier kernel is thus found by equating (11) to the sum of (14) and (17). However, before doing this the substitution $\Delta x = x-x_o$ and $\Delta y = y-y_o$ or $\Delta\vec{r}_t = \vec{r}_t-\vec{r}_{t_o}$ is first made in (17); then the equating of terms yields

$$\lim_{n\to\infty} \frac{1}{(2\pi)^{s_n}} \int\cdots\int \tilde{J}_q(\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots,\beta_n)\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n)\, d\beta_1 d\vec{\beta}_2\cdots d\beta_n$$

$$= 2H_o \exp(-j\vec{k}_{i_t}\cdot\vec{r}_t)\left[C_z^q + j\,C_x^q\,\partial_{k_{2x}} + j\,C_y^q\,\partial_{k_{2y}}\right]\tilde{p}_1\,(k_1-k_{iz},\vec{k}_2,\vec{k}_3,\cdots)$$

$$+ \lim_{n\to\infty} \frac{1}{(2\pi)^{2+s_n}} \int\cdots\int \left\{ j\,\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta x}\,\partial_{k_{2x}} + j\,\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta y}\,\partial_{k_{2y}} + \tilde{g}_\zeta(\Delta\vec{r}_t,\beta_o)\right\}$$

$$\cdot\ \tilde{J}_q(\vec{r}_t-\Delta\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)\,\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,\vec{k}_n,-\vec{\beta}_n;\Delta\vec{r}_t)$$

$$\cdot\ d\Delta\vec{r}_t\; d\beta_o\; d\beta_1\cdots d\vec{\beta}_n \tag{20}$$

Since all the three terms in (20) must exhibit the same dependence upon $\vec{r}_t$ and only $\tilde{J}_q$ and $\exp(-j\vec{k}_{i_t}\cdot\vec{r}_t)$ are functions of $\vec{r}_t$, (20) implies that $\tilde{J}_q$ may be written in the following factored form;

$$\tilde{J}_q(\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots) = j_q(\beta_1,\vec{\beta}_2,\cdots)\exp(-j\,\vec{k}_{i_t}\cdot\vec{r}_t) \tag{21}$$

Substituting this result in (20) and rearranging terms produces the following integral equation for $j_q(\beta_1,\vec{\beta}_2,\cdots)$;

$$\lim_{n\to\infty} \frac{1}{(2\pi)^{s_n}} \int\cdots\int j_q(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)\left[\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n)\right.$$

$$\left. -\ \Gamma_2(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n)\right]d\beta_1\; d\vec{\beta}_2\cdots d\vec{\beta}_n$$

$$= 2H_o\left[C_z^q + j\,C_x^q\,\partial_{k_{2x}} + j\,C_y^q\,\partial_{k_{2y}}\right]\tilde{p}_1\,(k_1-k_{iz},\vec{k}_2,\vec{k}_3,\cdots) \tag{22}$$

where

$$\Gamma_2(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n) = \frac{1}{(2\pi)^2}\iint\left\{j\,\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta x}\,\partial_{k_{2x}} + j\,\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta y}\,\partial_{k_{2y}}\right.$$

$$\left. + \tilde{g}_\zeta(\Delta\vec{r}_t,\beta_o)\right\}\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,\vec{k}_n,-\vec{\beta}_n:\Delta\vec{r}_t)$$

$$\cdot\exp(j\,\vec{k}_{i_t}\cdot\Delta\vec{r}_t)\,d\Delta\vec{r}_t\,d\beta_o \tag{22a}$$

Equations (21) and (22) provide the equations for determining the x and y components of the stochastic transform of the current subject to the neglect of the coupling term in (4). The inclusion of the coupling term is not difficult and, in fact, can be done by inspection in view of its similarity to those terms already included in (22). Thus, the complete result is as follows for $q = x$ or y and $\Delta q = \Delta x$ or Δy;

$$\lim_{n\to\infty}\frac{1}{(2\pi)^{s_n}}\int\cdots\int j_q(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)\Big[\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n)$$

$$-\Gamma_2(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n)\Big]\,d\beta_1\,d\vec{\beta}_2\cdots d\vec{\beta}_n - \lim_{n\to\infty}\frac{1}{(2\pi)^{s_n}}\int\cdots\int j_x(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)$$

$$\cdot\,\Gamma_x(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n)d\beta_1 d\vec{\beta}_2\cdots d\vec{\beta}_n - \lim_{n\to\infty}\frac{1}{(2\pi)^{s_n}}\int\cdots\int j_y(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)$$

$$\cdot\,\Gamma_y(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n)d\beta_1 d\vec{\beta}_2\cdots d\vec{\beta}_n = 2H_o\left[C_z^q+jC_x^q\partial_{k_{2x}}+jC_y^q\partial_{k_{2y}}\right]\tilde{p}_1(k_1-k_{iz},\vec{k}_2,\vec{k}_3,\cdots) \tag{23}$$

where

$$\Gamma_x(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,\vec{k}_n) = -\frac{1}{(2\pi)^2}\iint\frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta q}\left[j\partial_{k_{2x}}+j\partial_{\beta_{2x}}\right]\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,$$

$$\vec{k}_n,-\vec{\beta}_n;\Delta\vec{r}_t)\exp(j\vec{k}_{i_t}\cdot\Delta\vec{r}_t)d\Delta\vec{r}_t d\beta_o \tag{23a}$$

$$\Gamma_y(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,\cdots,\vec{\beta}_n,k_n) = -\frac{1}{(2\pi)^2}\iint \frac{\partial\tilde{g}(\Delta\vec{r}_t,\beta_o)}{\partial\Delta q}\left[j\partial_{k_{2y}}+j\partial_{\beta_{2y}}\right]\tilde{p}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,$$

$$\vec{k}_n,-\vec{\beta}_n;\Delta\vec{r}_t)\exp(j\vec{k}_{i_t}\cdot\Delta\vec{r}_t)d\Delta\vec{r}_t d\beta_o \quad (23b)$$

The z-component of the stochastic transform of the current is obtained by averaging (3) weighted by the Fourier kernel. This operation leads to the following;

$$\lim_{n\to\infty}\frac{1}{(2\pi)^{1+a_n}}\int\cdots\int\left\{j_z(\beta_1,\vec{\beta}_2,\cdots,\beta_n)-j\frac{\partial j_x(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)}{\partial\beta_{2x}}-j\frac{\partial j_y(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)}{\partial\beta_{2y}}\right\}$$

$$\cdot\ \tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n)\ d\beta_1\ d\vec{\beta}_2\cdots d\vec{\beta}_n = 0 \quad (24)$$

Since this result must hold for all values of $k_1,\vec{k}_2,\cdots$, it follows that

$$j_z(\beta_1,\vec{\beta}_2,\cdots) = j\left[\frac{\partial j_x(\beta_1,\vec{\beta}_2,\cdots)}{\partial\beta_{2x}}+\frac{\partial j_y(\beta_1,\vec{\beta}_2,\cdots)}{\partial\beta_{2y}}\right] \quad (25)$$

Discussion

One of the first results of the stochastic transform approach is illustrated in (21); that is, the dependence of the transformed current upon the transverse coordinates x and y is identical to that of the incident plane wave. An important implication of this result is that the average scattered field (obtained by substituting (21) in (8) and this result in the average of (5)) is always specular. That is, the average scattered field is non-zero only for $k_{sx} = k_{ix}$ and $k_{sy} = k_{iy}$. It should be noted that the specular nature of the average scattered field is a general and an exact result for homogeneous surface statistics. However, if the surface has nonhomogeneous statistics then $\langle\vec{E}_s\rangle$ will not, in general, be specular because the joint two-point probability density function will no longer depend only on the difference coordinates $\Delta\vec{r}_t$.

The specular nature of the average scattered field was previously demonstrated by DeSanto [5] using the diagrammatic approach.

It is interesting to compare (22)* with the results obtained from the diagrammatic technique [6]. In the case of the latter approach one obtains an infinite hierarchy of one-dimensional singular integral equations of the second kind. In order to obtain the integral equations in the hierarchy, one must be relatively familiar with the use of diagram techniques to form the appropriate "mass operator" which is related to both the source term and the kernel in the resulting integral equations. Furthermore, since at a given level of the hierarchy, the resulting integral equaton depends in part on the solution of the next lower level integral equation, the approach has practical numerical limitations. Finally, it is somewhat difficult to see how certain statistical features of the surface impact the hierarchy of equations. By contrast, (22) is a single integral equation of the first kind involving an infinite number of variables. Because of the infinite range of integration, the integral equation is of the singular type with respect to all variables [10, pg. 160]. Thus, the fact that (22) is an integral equation of the first kind should not cause too great a problem. Obviously, (22) cannot be solved numerically for much beyond three variables $(k_1,\vec{k}_2)$ unless the kernel has a very special behavior with respect to the higher dimensions. Based upon these points of comparison, (22) exhibits no clear advantages with respect to the diagrammatic approach except that it is more compact.

There is, however, one aspect of (22) which is extremely useful and does not appear to be obviously present in the diagrammatic technique. The importance of <u>all</u> the higher order surface height derivatives is <u>explicitly</u> contained

*The remainder of this section will deal with the scalar formulation for j_q because of its more compact form. The concepts and developments apply equally well to the vector equation (23) however.

in the two-point joint characteristic function $\tilde{\tilde{p}}_2(\cdot)$ in (22a). It is therefore a reasonably straightforward task to determine the conditions under which a certain surface height derivative is no longer important in the determination of the stochastic Fourier transform of the current. The logic for accomplishing this task is as follows. Assume, for example, that one would like to know when the curvature, $\nabla^2\zeta$, and all higher order derivatives have no significant impact on the current. If they are not important then $J_q(\vec{r}_t,\zeta,\nabla\zeta,\cdots)$ will be essentially independent of $\nabla^2\zeta,\nabla^3\zeta,\cdots$ so that $j_q(\beta_1,\vec{\beta}_2,\vec{\beta}_3,\cdots)=j_q(\beta_1,\vec{\beta}_2)(2\pi)^{s_\infty-3}\prod_{n=3}^{\infty}\delta(\vec{\beta}_n)$ because the transform of a constant is a delta distribution. Substituting this result in (22) yields;

$$\frac{1}{(2\pi)^3}\int\cdots\int j_q(\beta_1,\vec{\beta}_2)\,\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\vec{k}_3,\cdots,\vec{k}_n)\Big\{1$$

$$-\Gamma_2(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,0,\vec{k}_3,\cdots,0,\vec{k}_n)\Big/\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\beta_2,\vec{k}_3,\cdots,\vec{k}_n)\Big\}\,d\beta_1 d\vec{\beta}_2$$

$$=2H_o\Big[C_z^q + j\,C_x^q\partial_{k_{2x}} + j\,C_y^q\partial_{k_{2y}}\Big]\tilde{p}_1(k_1-k_{1z},\vec{k}_2,\vec{k}_3,\cdots) \qquad (26)$$

In order to make things explicit, let the stochastic variables $\nabla^2\zeta,\nabla^3\zeta,\cdots$ be statistically independent of ζ and $\nabla\zeta$. Then the single-point characteristic function may be factored as follows;

$$\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\vec{k}_3,\cdots)=\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2)\,\tilde{p}_1(\vec{k}_3,\vec{k}_4,\cdots)$$

where $\tilde{p}_1(\cdot)$ is the single-point characteristic function for the indicated variables. Substituting this identity in (26) and dividing by $\tilde{p}_1(\vec{k}_3,\vec{k}_4,\cdots)$ yields

$$\frac{1}{(2\pi)^3}\int\cdots\int j_q(\beta_1,\vec{\beta}_2)\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2)\Big\{1$$

$$-\Gamma_2(\beta_1,k_1,\vec{\beta}_2,\vec{k}_2,0,\vec{k}_3,\cdots,0,\vec{k}_n)/\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2)\,\tilde{p}_1(\vec{k}_3,\vec{k}_4,\cdots,k_n)\Big\}\,d\beta_1 d\vec{\beta}_2$$

$$=2H_o\Big[C_z^q + j\,C_x^q\partial_{k_{2y}} + j\,C_y^q\partial_{k_{2y}}\Big]\tilde{p}_1(k_1-k_{1z},\vec{k}_2) \qquad (27)$$

The right side of (27) is independent of $\vec{k}_3,\vec{k}_4,\cdots$ whereas the left side still depends on the variables through the $\Gamma_2/\tilde{p}_1$ term. Since this is an impossible situation, one examines the $\Gamma_2/\tilde{p}_1$ term to determine the conditions for which it is essentially (but not trivially) independent of $\vec{k}_3,\vec{k}_4,\cdots$. Thus, the stochastic Fourier transform technique provides a very orderly procedure for estimating the importance of the surface height derivatives. This result appears to be one of the most important advantages of this technique.

Limiting Cases

In the previous section the stochastic Fourier transform approach has been introduced, developed, and one of its merits has been discussed. However, the real power implicit in the technique results from its ability to provide exact results for two very important limiting cases. In order to be specific, the randomly rough surface is assumed to be a jointly Gaussian process; [11] provides details on the form of $\tilde{p}_1$ and $\tilde{\tilde{p}}_2$ for a Gaussian process. Such an assumption is not essential to the following development but it does facilitate comparison with previous results. Furthermore, it will be assumed that the stochastic surface height has a well defined spatial correlation function $R(\Delta\vec{r}_t) = \langle\zeta(\vec{r}_{t_1})\zeta(\vec{r}_{t_1}+\Delta\vec{r}_t)\rangle$ which decays with increasing $|\Delta\vec{r}_t|$ such that the concept of a correlation length (ℓ) has meaning.

Gently Undulating Surface ($\ell \gg \lambda_o$)

The first limiting case to be considered is that of a surface of arbitrary height but so gently undulating that none of the surface height derivatives ($\nabla^n\zeta, n=1,2,\cdots$) are important to the determination of the current. For such a surface, the correlation length is necessarily much larger than the electromagnetic wavelength ($\ell \gg \lambda_o$).

Using the technique developed in the previous section it is possible to determine the conditions under which the height derivatives are not important. To shorten matters somewhat, the technique will be demonstrated by considering only surface slopes. That is, $<(\nabla^n\zeta)^2>$ for n=2,3,··· are assumed to be vanishingly small so that the single and two-point joint probability density functions approach centered delta distributions with respect to these variables. Consequently, the joint characteristic functions $\tilde{p}_1$ and $\tilde{\tilde{p}}_2$ in (23) become independent of $\vec{\beta}_n$ and $\vec{k}_n$ for n=2,3,··· and the transform of the current may be written as follows;

$$j_q(\beta_1,\vec{\beta}_2,\vec{\beta}_3,\cdots) = j_q(\beta_1,\vec{\beta}_2)(2\pi)^{s_\infty-3}\prod_{i=3}^{\infty}\delta(\vec{\beta}_i) \tag{28}$$

This result is merely a restatement of the fact that the current, under these conditions, is independent of the surface curvatures $(\nabla^2\zeta)$, rate of change of curvatures $(\nabla^3\zeta)$, etc. To determine <u>when the slopes $(\nabla\zeta)$ may be ignored</u>, $\tilde{p}_1$ is factored out of Γ_2, Γ_x, and Γ_y in (23), $j_q(\beta_1,\vec{\beta}_2)$ is assumed to be of the following form

$$j_q(\beta_1,\vec{\beta}_2) = j_q(\beta_1)\delta(\vec{\beta}_2)(2\pi)^2 \tag{29}$$

and the residual dependence of $\Gamma_2/\tilde{p}_1$, $\Gamma_x/\tilde{p}_1$, and $\Gamma_y/\tilde{p}_1$ upon $\vec{k}_2$ is examined after dividing both sides of (23) by $\tilde{p}_1(\vec{k}_2)$. These operations lead to the following simplified form for (23);

$$\frac{1}{2\pi}\int j_q(\beta_1)\tilde{p}_1(k_1-\beta_1)\left\{1-\Gamma_2(\beta_1,k_1,\vec{k}_2)/\tilde{p}_1(k_1-\beta_1,\vec{k}_2)\right\}d\beta_1 - \frac{1}{2\pi}\int j_x(\beta_1)\tilde{p}_1(k_1-\beta_1)$$

$$\cdot\Gamma_x(\beta_1,k_1,\vec{k}_2)/\tilde{p}_1(k_1-\beta_1,\vec{k}_2)d\beta_1 - \frac{1}{2\pi}\int j_y(\beta_1)\tilde{p}_1(k_1-\beta_1)\Gamma_y(\beta_1,k_1,\vec{k}_2)/\tilde{p}_1(k_1-\beta_1,\vec{k}_2)d\beta_1$$

$$= 2H_o\left[C_z^q - j<\zeta_x^2>k_{2x}C_x^q - j<\zeta_y^2>k_{2y}C_y^q\right]\tilde{p}_1(k_1-k_{iz}) \tag{30}$$

where

$$\frac{\Gamma_2}{\tilde{p}_1} = \frac{1}{(2\pi)^2}\iint\left\{j\frac{\partial\tilde{g}}{\partial\Delta x}\left[-<\zeta_x^2>k_{2x}+(\beta_o-\beta_1)R_x\right] + j\frac{\partial\tilde{g}}{\partial\Delta y}\left[-<\zeta_y^2>k_{2y}+(\beta_o-\beta_1)R_y\right]+\tilde{g}_\zeta\right\}$$

$$\cdot\ \chi\exp(j\vec{k}_{i_t}\cdot\Delta\vec{r}_t)d\Delta\vec{r}_t d\beta_o \tag{30a}$$

$$\frac{\Gamma_x}{\tilde{p}_1} = -\frac{1}{(2\pi)^2}\iint j\,\frac{\partial\tilde{g}}{\partial\Delta q}\left\{-\left[<\zeta_x^2>+R_{xx}\right]k_{2x}+(2\beta_o-\beta_1-k_1)R_x-R_{xy}k_{2x}\right\}\chi\exp(j\vec{k}_{i_t}\cdot\Delta\vec{r}_t)\,d\Delta\vec{r}_t\,d\beta_o \tag{30b}$$

$$\frac{\Gamma_y}{\tilde{p}_1} = -\frac{1}{(2\pi)^2}\iint j\,\frac{\partial\tilde{g}}{\partial\Delta q}\left\{-\left[<\zeta_y^2>+R_{yy}\right]k_{2y}+(2\beta_o-\beta_1-k_1)R_y-R_{xy}k_{2y}\right\}\chi\exp(j\vec{k}_{i_t}\cdot\Delta\vec{r}_t)\,d\Delta\vec{r}_t\,d\beta_o \tag{30c}$$

and

$$\chi = \exp\left\{-\left[<\zeta^2>-R\right](\beta_o-\beta_1)(\beta_o-k_1) + (\beta_o-\beta_1)(k_{2x}R_x+k_{2y}R_y)\right\} \tag{30d}$$

In the above expressions $<\zeta_x^2>$ and $<\zeta_y^2>$ are the variances of the slopes in the x and y directions while $R_x = \partial R/\partial\Delta x$, $R_y = \partial R/\partial\Delta y$, etc.

Consistent with the assumption in (29), $j_q(\beta_1)$ is to be determined from (30) and the resulting solution must be independent of the slope statistics and $\vec{k}_2$. For arbitrary slope variances this is not possible and it is necessary to examine the conditions under which the slope dependent terms in (30) can be made small. The terms on the right hand side of (30) which depend linearly upon $\vec{k}_2$ can be made small by taking $<\zeta_x^2> \ll 1$ and $<\zeta_y^2> \ll 1$. If the slope variances are small, the correlation function may be approximated as follows;

$$R(\Delta\vec{r}_t) \simeq <\zeta^2> - \frac{1}{2}<\zeta_x^2>(\Delta x)^2 - \frac{1}{2}<\zeta_y^2>(\Delta y)^2 \tag{31}$$

because $<(\nabla^n\zeta)^2>$ for $n=2,3,\cdots$ have already been assumed to be vanishingly small. Also, the coordinate system has been oriented such that there are no $\Delta x\cdot\Delta y$ terms in (31). In view of (31), the terms in (30b) and (30c) which depend linearly on k_{2x} and k_{2y}, respectively, vanish. The exponential in (30d), using (31), may be written as follows;

$$\chi = \exp\left\{-\frac{1}{2}\left[<\zeta_x^2>(\Delta x)^2 + <\zeta_y^2>(\Delta y)^2\right](\beta_o-\beta_1)(\beta_o-k_1)\right\}$$

$$\cdot\sum_{n=o}^{\infty}\frac{(-1)^n(\beta_o-\beta_1)^n\left[<\zeta_x^2>k_{2x}\Delta x + <\zeta_y^2>k_{2y}\Delta y\right]^n}{n!} \tag{32}$$

The terms in (30a)-(30c) which have R_x or R_y multiplying the Green's function derivatives may be lumped with the series expansion in (32). The $\Delta\vec{r}_t$-integration in (30a)-(30c) may now be treated as a convolution of the $\Delta\vec{r}_t$-Fourier transform of (32) with the transform of the Green's function derivatives. The transform of the Gaussian factor in (32) is

$$\frac{1}{2\pi(\beta_o-\beta_1)(\beta_o-k_1)\sqrt{<\zeta_x^2><\zeta_y^2>}} \exp\left\{\frac{1}{2(\beta_o-\beta_1)(\beta_o-k_1)}\left[-\frac{k_{ix}^2}{<\zeta_x^2>}-\frac{k_{iy}^2}{<\zeta_y^2>}\right]\right\} \tag{32a}$$

For small slope variances (32a) behaves as a delta distribution which combs the transform of the Green's function derivatives out of convolution. The terms in the power series in (32) give rise to n^{th} derivatives of the delta distribution which, in turn, yields derivatives of the transform of the Green's functions. However, because of the slope variances in the power series in (32), the n^{th} derivative of the transform of the Green's functions will be multiplied by a slope variance raised to the n^{th} power which is vanishingly small. Thus, the only term left in (30a)-(30d) is

$$\Gamma_2/\tilde{p}_1 \approx \frac{1}{(2\pi)^2}\iint \tilde{g}_\zeta \exp(j\vec{k}_{it}\cdot\Delta\vec{r}_t)\, d\Delta\vec{r}_t d\beta_o$$

However since the $\Delta\vec{r}_t$-Fourier transform of $\tilde{g}_\zeta$ is an odd function of β_o, $\Gamma_2/\tilde{p}_1$ is identically zero and (30) becomes

$$\frac{1}{2\pi}\int j_q(\beta_1)\tilde{p}_1(k_1-\beta_1)d\beta_1 \approx 2C_z^q\,\tilde{p}_1(k_1-k_{iz}) \tag{33}$$

so that $j_q(\beta_1) = 2H_o\, C_z^q\, \delta(\beta_1-k_{iz})(2\pi)$ or

$$j_q(z) = 2H_o\, C_z^q \exp(-jk_{iz}\,\zeta) \tag{34}$$

which is the scalar single scattering physical optics result. With $k_1 = k_{sz} = -k_{iz}$, (33) shows that the average field scattered by the surface is attenuated from its flat plane value by the factor $\tilde{p}_1(-2k_{iz}) = \exp(-2k_o^2<\zeta^2>\cos^2\theta)$. This is the classical result which simply says that if the surface is sufficiently smooth, the average scattered field appears to be due to a randomly elevated plane. The important point to note from this development is that for a very gently undulating surface which has a correlation length which is large compared to λ_o, scalar single scattering or physical optics is applicable.

Uniformly Rough Surface ($\ell << \lambda_o$)*

A much more interesting limiting case is provided by surfaces which have arbitrary height but a correlation length which is small compared to λ_o. Such surfaces necessarily have a great deal of their height variance contributed by undulation frequencies the order of and greater than k_o, the electromagnetic wavenumber. Consequently, there is absolutely no reason to expect physical optics to be an adequate approximation for the scattering process and, indeed, this is found to be the case.

Before proceeding, it is worthwhile pointing out why it is the ratio of the correlation length to λ_o that is a critical parameter. Departures from single scattering are caused by the Γ_2, Γ_x, and Γ_y terms in (23). These terms, as illustrated in (22a), (23a), and (23b), are determined by an integration over $\Delta\vec{r}_t$. The quantities which appear in the $\Delta\vec{r}_t$-integrands are dependent upon λ_o and the correlation length as scale parameters. Thus, these parameters determine whether it is the electromagnetic functions or the statistical functions which dominate the $\Delta\vec{r}_t$-integration.

When the correlation length is much smaller than λ_o, the statistical functions in (23) vary much more rapidly with $\Delta\vec{r}_t$ than the electromagnetic functions. In fact, the statistical functions go from completely correlated to completely

*It is difficult to describe a surface having $\ell << \lambda_o$ with a few short words. "Uniformly rough" is appropriate when the correlation function is a Gaussian function of the form $\exp(-|\Delta\vec{r}_t|^2/\ell^2)$. In this case the surface height spectrum is also Gaussian and if $\ell << \lambda_o$ then the roughness is spread nearly uniformly throughout all spatial frequencies $\leq k_o = 2\pi/\lambda_o$.

decorrelated over a region of essentially zero measure when compared to the scale of variation of the electromagnetic functions. Consequently, the statistical functions in (23) may be replaced by their values for complete decorrelation, i.e. $|\Delta\vec{r}_t| \to \infty$, with no appreciable error. This implies that the correlation function and all of its higher order derivatives are set to zero in (22a), (23a), and(23b). Using this approximation and the fact that decorrelation implies statistical independence for a Gaussian process, $\hat{p}_2$ in the expressions for Γ_2, Γ_x, and Γ_y simplifies to the following form [11];

$$\tilde{\tilde{p}}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots,\vec{k}_n,-\vec{\beta}_n) = \exp\left[-<\zeta^2>(\beta_o-\beta_1)(\beta_o-k_1)\right]\tilde{p}_1(k_1-\beta_1)$$

$$\cdot\prod_{i=2}^{n}\tilde{p}_1(\vec{k}_i)\tilde{p}_1(-\vec{\beta}_i)$$

where $\tilde{p}_1(\vec{k}_i)$ and $\tilde{p}_1(-\vec{\beta}_i)$ are the characteristic functions for $\nabla^i\zeta$. Substituting these simplifications in (23) and noting that

$$\tilde{p}_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots,\vec{k}_n-\vec{\beta}_n) = \tilde{p}_1(k_1-\beta_1)\prod_{i=2}^{n}\tilde{p}_1(\vec{k}_i-\vec{\beta}_i) \quad ,$$

it is found that $j_q(\beta_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)$ has the following dependence upon $\vec{\beta}_i, i=3,4,\cdots$;

$$j_q(\vec{\beta}_1,\vec{\beta}_2,\cdots,\vec{\beta}_n) = j_q(\beta_1,\vec{\beta}_2)(2\pi)^{s_n-3}\prod_{i=3}^{n}\delta(\vec{\beta}_i) \tag{35}$$

Using this in (23) and dividing out the common dependence upon $\prod_{i=2}^{n}\tilde{p}_1(\vec{k}_i)$ results in the following two coupled equations for the current components;

$$\frac{1}{(2\pi)^3}\iint\tilde{p}_1(k_1-\beta_1)\tilde{p}_1(-\vec{\beta}_2)\Big\{j_x(\beta_1,\vec{\beta}_2)\exp\left[<\zeta_x^2>k_{2x}\beta_{2x} + <\zeta_y^2>k_{2y}\beta_{2y}\right]$$

$$-j_x(\beta_1,\beta_2)\int\left[j<\zeta_x^2>\beta_{2x}\tilde{\tilde{g}}_x - j<\zeta_y^2>k_{2y}\tilde{\tilde{g}}_y + \tilde{\tilde{g}}_\zeta\right]\overline{\chi}\,d\beta_o$$

$$-j_y(\beta_1,\vec{\beta}_2)\int\left[j<\zeta_x^2>(k_{2y}+\beta_{2y})\tilde{\tilde{g}}_x\right]\overline{\chi}\,d\beta_o\Big\}\,d\beta_1 d\vec{\beta}_2$$

$$= 2H_o\left[C_x^x - j<\zeta_y^2>k_{2y}C_y^x\right]\tilde{p}_1(k_1-k_{1z}) \tag{36a}$$

$$\frac{1}{(2\pi)^3}\iint \tilde{p}_1(k_1-\beta_1)\tilde{p}_1(-\vec{\beta}_2)\Big\{ j_y(\beta_1,\vec{\beta}_2)\exp\left[<\zeta_x^2>k_{2x}\beta_{2x} + <\zeta_y^2>k_{2y}\beta_{2y}\right]$$

$$- j_y(\beta_1\vec{\beta}_2)\int\left[- j<\zeta_x^2>k_{2x}\tilde{\tilde{\tilde{g}}}_x + j<\zeta_y^2>\beta_{2y}\tilde{\tilde{\tilde{g}}}_y + \tilde{\tilde{\tilde{g}}}_\zeta\right]\overline{\chi}\, d\beta_o$$

$$- j_x(\beta_1,\vec{\beta}_2)\int\left[j<\zeta_x^2>(k_{2x}+\beta_{2x})\tilde{\tilde{\tilde{g}}}_y\right]\overline{\chi}\, d\beta_o\Big\}\, d\beta_1 d\vec{\beta}_2 = 2H_o\left[C_z^y - j<\zeta_x^2>k_{2x}C_x^y\right]\tilde{p}_1(k_1-k_{iz}) \tag{36b}$$

where the triple tilde symbol over the Green's function derivatives denotes the three-dimensional Fourier transform with respect to Δx, Δy, and $\Delta\zeta$. The function $\overline{\chi}$ is given by

$$\overline{\chi} = (2\pi)^{-2}\exp[-<\zeta^2>(\beta_o-\beta_1)(\beta_o-k_1)] \quad .$$

Expanding the exponential factor containing k_{2x} and k_{2y} in a series and equating like powers of k_{2x} and k_{2y} on both sides of (36), it becomes clear that $j_q(\beta_1,\vec{\beta}_2)$ may be expressed as follows;

$$\begin{aligned} j_x(\beta_1,\vec{\beta}_2) &= \left\{ j_x^{(1)}(\beta_1)\delta(\beta_{2x})\delta(\beta_{2y}) + j_x^{(2)}(\beta_1)\delta(\beta_{2x})\delta'(\beta_{2y})\right\}(2\pi)^2 \\ j_y(\beta_1,\vec{\beta}_2) &= \left\{ j_y^{(1)}(\beta_1)\delta(\beta_{2x})\delta(\beta_{2y}) + j_y^{(2)}(\beta_1)\delta'(\beta_{2x})\delta(\beta_{2y})\right\}(2\pi)^2 \end{aligned} \tag{37}$$

Substituting these results in (36) and equating like coefficients of k_{2x} and k_{2y} yields the following four equations for the β_1-dependent parts of the transformed current;

$$\frac{1}{2\pi}\int j_q^{(1)}(\beta_1)\tilde{p}_1(k_1-\beta_1)\left\{1 - \frac{1}{(2\pi)^2}\int\tilde{\tilde{\tilde{g}}}_\zeta \exp\left[-<\zeta^2>(\beta_o-\beta_1)(\beta_o-k_1)\right]d\beta_o\right\}d\beta_1 = 2H_o C_z^q\tilde{p}_1(k_1-k_{iz}) \tag{38}$$

$$\frac{1}{2\pi}\int\tilde{p}_1(k_1-\beta_1)\Big\{ j_q^{(2)}(\beta_1) + \frac{j}{(2\pi)^2} j_q^{(1)}(\beta_1)\int\tilde{\tilde{\tilde{g}}}_{q'}\exp\left[-<\zeta^2>(\beta_o-\beta_1)(\beta_o-k_1)\right]d\beta_o$$

$$- \frac{j}{(2\pi)^2} j_{q'}^{(1)}(\beta_1)\int\tilde{\tilde{\tilde{g}}}_q \exp\left[-<\zeta^2>(\beta_o-\beta_1)(\beta_o-k_1)\right]d\beta_o\Big\}d\beta_1 = - j\, 2H_o C_{q'}^q\tilde{p}_1(k_1-k_{iz}) \tag{39}$$

where q and q' are paired as $q = x$, $q' = y$ and $q = y, q' = x$. Using (37) in (21), (5), and (8) shows that the part of the current which depends upon $\delta'(\beta_{2q})$ contributes nothing to the average scattered field which from (5) may be expressed as follows;

$$\langle E_s(k_{sz})\rangle = j\pi k_o \eta_o g(R_o)\delta(\vec{k}_{s_t} - \vec{k}_{i_t})[\hat{k}_s \times \hat{k}_s \times \vec{T}_q(k_{sz})] \tag{40}$$

where for $q = x$ or y,

$$T_q(k_{sz}) = \frac{1}{2\pi}\int j_q^{(1)}(k_1)\,\tilde{p}_1(k_{sz} - k_1)dk_1 \tag{41}$$

and $k_{sz} = -k_{iz}$ because of the specularity of the average scattered field. Substituting (41) in (38), combining $\tilde{p}_1(k_1 - \beta_1)$ with the exponential term [11], and noting that

$$\tilde{\tilde{\tilde{g}}}_\zeta(\beta_o) = j\,4\pi \lim_{\varepsilon\to 0}\frac{\beta_o}{\beta_o^2 - k_{iz}^2 + j\varepsilon}$$

yields the following integral equation for $T_q(k)$;

$$T_q(k) = 2H_o[\hat{q}\cdot(\hat{z}\times\hat{h})]\tilde{p}_1(k - k_{iz}) + \frac{j}{\pi}\int \lim_{\varepsilon\to 0}\left[\frac{\beta_o}{\beta_o^2 - k_{iz}^2 + j\varepsilon}\right]\tilde{p}_1(k - \beta_o)T_q(\beta_o)d\beta_o \tag{42}$$

Both (42) and some of the interim results leading to it have significant implications. Eqns. (35) and (37) imply that the product of the surface current and $\sqrt{1 + \zeta_x^2 + \zeta_y^2}$ may be divided into two parts with one part totally independent of the surface height derivatives, $\nabla^n\zeta$. It is tempting to declare that the other part depends linearly on one of the slope components since this would appear to follow from the $\delta'(\beta_{2q})$ in (37) and the Fourier transform relationship between J_q and j_q, see (11) and (21). However, it is not clear that such a Fourier inversion uniquely recovers the true stochastic nature of the current, i.e. the true dependence on $\vec{r}_t$. This simply says that one cannot, in general, go from j_q back to $\vec{J}_s\cdot\hat{q}$.

Using (38) and (39) and the fact that $C_y^x = -C_x^y$, it can be shown that

$$\int j_x^{(2)}(\beta_1)\tilde{p}_1(k_1 - \beta_1)d\beta_1 = -\int j_y^{(2)}(\beta_1)\tilde{p}_1(k_1 - \beta_1)d\beta_1 \tag{43}$$

and that $j_x^{(2)}(\beta_1)$ can be determined from knowledge of $T_x(k)$ according to the following;

$$\frac{1}{2\pi}\int j_x^{(2)}(\beta_1)\tilde{p}_1(k_1-\beta_1)d\beta_1 = -\,j\,2H_oC_y^x\,\tilde{p}_1(k_1-k_{iz}) + \frac{1}{\pi}\left(k_{iy}-k_{ix}\frac{C_z^y}{C_z^x}\right)$$

$$\cdot\int \lim_{\varepsilon\to 0}\left\{\frac{1}{\beta_o^2-k_{iz}^2+j\varepsilon}\right\}\tilde{p}_1(k_1-\beta_o)T_x(\beta_o)d\beta_o \qquad (44)$$

for $C_z^x \neq 0$. It is important to realize however that $j_q^{(2)}$ contributes nothing to the average scattered field. Discussion of the effect of $j_q^{(2)}$ on the mean square scattered field will be left to future investigations.

Equation (42) is very similar to the integral equation obtained by DeSanto [6] for this "uniformly rough" surface limit. It differs in that DeSanto obtains a β_o in the denominator of the kernel of the integral term rather than in the numerator as in (42). However, DeSanto's integral equation results from only the first term in his diagrammatic hierarchy of equations whereas (42) is exact. It is therefore not surprising that the two integral equations differ.

At the beginning of this section, it was stated that the gently undulating and uniformly rough surfaces are very important limiting cases. Although it is difficult to prove mathematically, there are strong physical agruments to suggest that these two cases provide lower and upper bounds on the average scattered field. That is, for a fixed height variance, $\langle\zeta^2\rangle$, the gently undulating surface gives a lower bound on the coherent field while the uniformly rough surface yields the upper bound. As noted previously, the gently undulating surface behaves as a randomly elevated plane which, in turn, puts all of the randomness of the surface in the phase of the scattered field. Upon averaging over an ensemble of scattered fields, the randomness of the phase leads to a maximum effect on the mean scattered field and, consequently, a minimum value. As the surface approaches the uniformly rough case, the randomness of the surface no longer appears totally in the phase of the scattered field due to multiple scattering on the surface. This is just a restatement of the fact that the physical optics approximation is not accurate for surface features having an undulation frequency which is small compared to k_o. Thus, once the correlation length of the surface exceeds the electromagnetic wavelength, the effects of the random height on the phase of the scattered field are minimized. Consequently, this should lead to a maximum coherent scattered field.

One final interesting facet of these two limiting cases is the points which they share in common. Both cases depend only on the $\partial g/\partial \zeta_o$ term in the integral in (10). For the gently undulating surface the contribution from this term is essentially zero. For the uniformly rough surface this term encompasses all of the multiple scattering that is important to the coherent or average scattered field. A second point of commonality is the fact that the currents which contribute to the average scattered field* are independent of all the surface height derivatives. These facts form the basis for an approximate approach to solving (42) which is detailed elsewhere [12].

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*For the gently undulating surface, this is the total current while for the uniformly rough surface it is only a portion of the current which contributes to $\langle \vec{E}_s \rangle$.

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New Results on Coherent Scattering From

Randomly Rough Conducting Surfaces

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Abstract

An approximate solution for the average field scattered by a perfectly conducting randomly rough surface having a correlation length much smaller than the electromagnetic wavelength is presented. The analysis is based on the use of a substitute surface which gives rise to the same describing equations as the true surface relative to the average scattered field. The substitute surface comprises large, non-overlapping, flat areas having random elevations with respect to the mean planar surface and arbitrary correlation between adjacent areas. The average scattered field is shown to depend upon the number of interacting areas and the surface roughness. For a given range of surface roughness there is a specific number of interacting areas which dominate the average scattered field. It is demonstrated how this number can be computed and how a continuous curve of average scattered field as a function of surface roughness is obtained. Of particular importance is the quantitative correspondence established in this paper between the surface roughness and the degree of multiple interaction on the rough surface.

Introduction and Summary

Single scattering approximations have served very nicely to illustrate some of the salient features of scattering from randomly rough surfaces [1]. There is, however, a lingering controversy as to how far these approximations can be extended [2]. The basic problem associated with establishing the range of validity of these approximations is that the next order correction is very difficult to obtain. A number of solutions have been presented which are formally exact but are practically limited to a two-term perturbation approximation [3,4]. The pioneering work of Zipfel and DeSanto [5] provided the first approximate correction to single scattering theory which did not require a small perturbation in the surface height. Subsequent numerical work by DeSanto and Shisha [6] showed that, for a Gaussian distributed surface, single scattering grossly underestimated the strength of the average scattered field for large surface roughness and small correlation length. Unfortunately, the detailed mathematics essential to obtaining this correction to the single scattering approximation did not lend itself to a straightforward physical interpretation [7].

Recently another approach to the rough surface scattering problem has been developed [8]. Apart from starting with the magnetic field integral equation, the unique feature of this approach was the conversion of averages to convolutions in stochastic transform space. The purpose of this new approach was not to duplicate the excellent work of Zipfel and DeSanto but rather to obtain a solution which clearly illustrated the range of validity of approximations necessary to simplify the full-blown problem. One of the results of this effort was the derivation of an exact compact integral equation for the average scattered field in the case of a surface having a correlation length much smaller than the electromagnetic wavelength. This equation was similar to the approximate integral equation obtained by Zipfel and DeSanto [5]. In [8] it was shown that the average field scattered from a randomly rough perfectly conducting surface was attenuated from its flat surface value by the quantity $\vec{T}(k = k_{sz} = -k_{iz})$ where k_{sz} is the z-component of the vector wavenumber in the scatter direction and k_{iz} is the z-component of the vector wavenumber in the incidence direction. The condition $k_{sz} = -k_{iz}$ results from the specularity of the average scattered field. For Gaussian surfaces having a correlation length (ℓ) much smaller than the electromagnetic wavelength (λ_o), the exact equation for $T_q(k)$ was found to be the following;

$$T_q(k) = 2H_o[\hat{q}\cdot(\hat{z}\times\hat{h})]\,\tilde{p}_1(k-k_{iz}) + \frac{j}{\pi}\int \lim_{\varepsilon\to o}\left[\frac{\beta_o}{\beta_o^2-k_{iz}^2+j\varepsilon}\right]\tilde{p}_1(k-\beta_o)\,T_q(\beta_o)\,d\beta_o \quad (1)$$

where H_o and $\hat{h}$ are, respectively, the amplitude and polarization of the magnetic field incident upon the surface. In (1) q is either x or y and $\tilde{p}_1$ is the Fourier transform of the surface height probability density function.

A slightly different form of this equation (with β_o in the denominator of the bracketed term rather than in the numerator) was solved numerically by DeSanto and Shisha [6] and it would appear to be a straightforward task to apply their technique to (1). However, this writer has some concerns about using the same technique to numerically solve (1). First, although the difference between (1) and the equation solved in [6] may appear to be minor, it is likely that the difference may significantly alter the convergence properties of the integral in (1). This, in turn, leads to the distinct possibility of singularities in $T_q(k)$ for $k \neq -k_{iz}$ which could not be accounted for using the quadratic spline technique employed in [6]. In short, the numerical solution of (1) may require a great deal more in-depth analysis than presented in [6].

Finally, there is a very practical limitation contained in a numerical solution of (1). Assuming that a suitable technique can be devised for solving (1) numerically, what one ends up with are numbers! While these numbers are certainly important, they add nothing to our understanding of the complex process of multiple scattering. If real progress is to be made in moving beyond our present knowledge, it must come from a clear understanding of the physics of the problem and not just numbers. Consequently, considering the potential difficulties associated with a numerical solution of (1) along with the limited insight available from such results, there is a definite need for an alternate solution of this problem. The purpose of this paper is to present one such alternate approximate solution.

The cornerstone of the new solution is the previously derived [8] similarity between the stochastic Fourier transform of the total current induced on a gently undulating surface ($\ell >> \lambda_o$) and the <u>part</u> of the stochastic Fourier transform of the current induced on a uniformly rough surface ($\ell << \lambda_o$) which is responsible for the coherent scattered field. That is, both of the transformed quantities have the same dependence on all surface height derivative statistics and they satisfy exactly the same reduced form of the transformed magnetic field integral equation. This similarity suggests that if the gently undulating surface ($\ell >> \lambda_o$) could be altered so that, from a mathematical point of view, the case of $\ell << \lambda_o$ could be addressed then this altered surface could be <u>substituted</u> for the uniformly rough surface●. Furthermore, if this substitute surface is such that its scattering properties with respect to the average field could be determined then these results would apply also to the uniformly rough surface. The key to success in such an approach is assuring that the substitute problem can be solved more easily than the original problem and that the <u>results</u> are in complete agreement with the physics of the problem.

The transformation of the gently undulating surface ($\ell >> \lambda_o$) into a substitute surface for the uniformly rough surface ($\ell << \lambda_o$) proceeds as follows. Because of the smallness of the surface slopes, curvatures, etc., the gently undulating surface may be approximated by an infinite number of large

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●It is assumed that the original similarity between the transformed currents is invariant under the alteration process.

non-overlapping conducting areas which are flat (non-inclined) but randomly elevated with respect to the $z = 0$ plane. The current induced on any one flat area is, however, determined by only a finite (but unknown) number of neighboring flat areas. An exact determination of the current on the rough surface, in this approximation, is possible and it is found to depend upon the relative elevations of the non-overlapping areas in a rather simple and physically meaningful manner. The complex amplitude of the average scattered field is obtained by averaging the product of the random height dependent part of the current and the Fourier kernel $\exp(j\, k_{sz}\, \zeta)$ where k_{sz} is the z-component of the wavenumber vector in the scattering direction and ζ is the random surface height.

In order to accomplish the field averaging one must have knowledge of the joint probability density function for the heights of the electromagnetically important neighboring areas. This, in turn, raises the question of the degree of correlation between these areas. Thus, it is at this point that total decorrelation may be inserted to complete the similitude between the substitute surface and uniformly rough surface, i.e. since $\ell << \lambda_o$ the uniformly rough surface may be approximately characterized by $\ell \approx 0$ or complete decorrelation. The substitute surface therefore comprises an infinite number of large non-overlapping conducting areas which are flat but randomly elevated and uncorrelated with each other.

In the case of surfaces for which decorrelation implies statistical independence (such as a Gaussian) the final result reduces to one integration which can be accomplished numerically. Not surprisingly, the average scattered field is found to depend upon the number of areas or regions on the rough surface allowed to interact with the point in question. For a fixed number of interacting areas, there is a specific range of roughness or Rayleigh parameter for which the average scattered field is a maximum. As the roughness decreases toward zero, the number of interacting areas on the surface which maximize the scattered field also approaches zero. Conversely, as the roughness increases the number of interacting areas necessary to maximize the average scattered field increases. In terms of the physics of the scattering process, these results have a very clear meaning. In essence the solution comprises a discrete approximation to the fact that, as the surface roughness increases, the current at any point on the surface is dependent upon an increasing area of the surrounding surface due to multiple interaction and not just the behavior of the surface in the immediate vicinity of the point in question. This result represents progress in the understanding of scattering from rough surfaces in that it is the first time that the complicated mathematics of mutual interaction have been simplified to the point where the result can be put into one-to-one correspondence with the physics of the process.

Numerical results for the strength of the average or coherent scattered field are presented for the case of a Gaussian distributed surface. These results show a significantly stronger coherent power for large surface roughness or Rayleigh parameter than is predicted by single scatter or the physical optics approximation. The apparent reason for the failure of the single scatter approximation is as follows. The single scatter or physical optics approximation attributes the effect of the surface roughness to a phase modification which, when averaged, tends to maximize the effect of the roughness. For $\ell << \lambda_o$, the solution obtained in this paper shows that mutual interaction gives rise to the same phase modification as obtained with single scatter plus an additional amplitude effect. However, this amplitude effect deemphasizes the influence of the surface roughness as predicted by the phase only approximation. This result also provides some insight into why iterative or "back and forth" multiple scattering attempts to solve the problem are successful only if the iterative series can at least be partially summed. That is, such attempts lead to series terms which contain, primarily, a phase perturbation; thus, no single term or any finite sum of terms can possibly describe the important amplitudes effects found in this paper.

A comparison with the results obtained by DeSanto and Shisha [6] is also presented. While there is agreement in predicting a greater average scattered field than single scattering for large surface roughness, there is also a difference in the detailed behavior. This is not surprising given that the DeSanto and Shisha result stems from only the first integral equation in an infinite hierarchy of integral equations.

Determination of the Surface Current

The rough surface provides the boundary between free-space and a perfectly conducting medium and it is assumed to be infinite in extent in the x and y coordinates of a conventional (x,y,z) coordinate system. The random surface height $\zeta(x,y)$ measured from the $z = 0$ plane is assumed to comprise a zero mean statistically homogeneous process with the mean surface equal to the $z = 0$ plane. The magnetic field integral equation [10, pg. 354] for the current $\vec{J}_s(\vec{r})$ induced on the surface at a point $\vec{r} = x\hat{x} + y\hat{y} + \zeta\hat{z}$ due to an incident magnetic field of the form $\vec{H}^i(\vec{r}) = H_o\hat{h}\exp(-j\,\vec{k}_i\cdot\vec{r})$ may be expressed in the following form [8];

$$\vec{J}(\vec{r}) = 2\vec{N}(\vec{r})\times\vec{H}^i(\vec{r}) + \frac{1}{2\pi}\int\left\{[\vec{N}(\vec{r})\cdot\nabla_o g]\,\vec{J}(\vec{r}_o) - [\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]\,\nabla_o g\right\}d\vec{r}_{t_o} \quad (2)$$

where $\vec{J}(\vec{r})$ is related to $\vec{J}_s(\vec{r})$ by

$$\vec{J}(\vec{r}) = \sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}\;\vec{J}_s(\vec{r})$$

and $\vec{N}(\vec{r}) = -\frac{\partial\zeta}{\partial x}\hat{x} - \frac{\partial\zeta}{\partial y}\hat{y} + \hat{z}$. The incident wavenumber vector $\vec{k}_i = k_{ix}\hat{x} + k_{iy}\hat{y} + k_{iz}\hat{z}$ is given by

$$\vec{k}_i = -k_o[\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}]$$

where θ is the angle of incidence with respect to the normal to the mean surface, ϕ is the azimuth angle measured from the x-axis, and $k_o = 2\pi/\lambda_o$ is the free-space wavenumber. The quantities $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are the x and y components of the surface slopes at the point $\vec{r}$ on the surface, $\vec{r}_{t_o} = x_o\hat{x} + y_o\hat{y}$, and $g(|\vec{r} - \vec{r}_o|)$ is proportional to the free-space Green's function, i.e.

$$g(|\vec{r} - \vec{r}_o|) = \frac{\exp(-j\,k_o|\vec{r} - \vec{r}_o|)}{|\vec{r} - \vec{r}_o|} \tag{3}$$

Using the fact that the current must be orthogonal to the surface normal, i.e. $\vec{N}(\vec{r}_o)\cdot\vec{J}(\vec{r}_o) = 0$, *leads to the following relationship between the vector com*ponents of the current;

$$J_z(\vec{r}_o) = (\partial\zeta_o/\partial x_o)\;J_x(\vec{r}_o) + (\partial\zeta_o/\partial y_o)\;J_y(\vec{r}_o) \tag{4}$$

Using (4), it can be shown that the term $[\vec{N}(\vec{r})\cdot\vec{J}(\vec{r}_o)]$ in (2) is proportional to the difference between the surface slopes at the points $\vec{r}$ and $\vec{r}_o$ on the surface. This term contributes nothing to that part of the current responsible for the coherent scattered field *for either* $\ell >> \lambda_o$ *or* $\ell << \lambda_o$ *[8] and* so (2) reduces to three scalar equations or, alternately, (4) and the following two expressions;

$$J_q(\vec{r}) = 2\hat{q}\cdot[\vec{N}(\vec{r}) \times \vec{H}^i(\vec{r})] + \frac{1}{2\pi}\int\left\{-\frac{\partial\zeta}{\partial x}\frac{\partial g}{\partial x_o} - \frac{\partial\zeta}{\partial y}\frac{\partial g}{\partial y_o} + \frac{\partial g}{\partial\zeta_o}\right\}J_q(\vec{r}_o)d\vec{r}_{t_o} \tag{5}$$

where $q = x$ or y and $\hat{q}$ is a unit vector in either of these directions. The slope dependent terms in (5) also can be shown to contribute nothing for either case [8] and (5) becomes

$$J_q(\vec{r}) = J_q^i(\vec{r}) + \frac{1}{2\pi} \int \frac{\partial g}{\partial \zeta_o} J_q(\vec{r}_o)\, d\vec{r}_{t_o} \tag{6}$$

for $q = x$ or y and $J_q^i(\vec{r}) = 2\hat{q}\cdot[\hat{z} \times \vec{H}^i(\vec{r})]$

The argument of the Green's function in (6) is $\sqrt{|\vec{r}_t - \vec{r}_{t_o}|^2 + (\zeta - \zeta_o)^2}$. Starting with the gently undulating surface ($\ell >> \lambda_o$) and using the fact that the surface curvatures and slopes are very small implies that the quantity $(\zeta - \zeta_o)$ is nearly a constant over a large range of $\vec{r}_{t_o}$. Thus, (6) may be approximated as follows;

$$J_q(\vec{r}) \approx J_q^i(\vec{r}) - \frac{1}{2\pi} \sum_{n=1}^{M} \int_{A_n} \frac{\partial g(\vec{r}_t - \vec{r}_{t_n}, \zeta - \zeta_n)}{\partial(\zeta - \zeta_n)} J_q(\vec{r}_n)\, d\vec{r}_{t_n} \tag{7}$$

where A_n is the area of the $z = 0$ plane over which $\zeta - \zeta_n$ is nearly constant and M+1 is the number of these areas which make up the entire $z = 0$ plane. To include the entire $z = 0$ plane in the integration in (7), M should be infinite; however this ignores the fact that only a finite area about the point $(\vec{r}_t, \zeta)$ is effective in determining $J_q(\vec{r}_t, \zeta)$. Consequently, M will be replaced by N which is assumed to be finite.

In essence (7) is a discrete approximation to (6); retention of the integration in (7) merely reflects the fact that $(\zeta - \zeta_n)$ is constant to some prescribed tolerance over an area A_n and not just a point. Defining $S_n(\vec{r}_{t_n})$ to be the support function for the n^{th} area, i.e. $S_n = 1$ for $\vec{r}_{t_n} \in A_n$ and zero otherwise, (7) may be rewritten as follows;

$$J_q(\vec{r}) \approx J_q^i(\vec{r}) - \frac{1}{2\pi} \sum_{n=1}^{N} \int_{-\infty}^{\infty} \frac{\partial g(\vec{r}_t - \vec{r}_{t_n}, \zeta - \zeta_n)}{\partial(\zeta - \zeta_n)} J_q(\vec{r}_n)\, S_n(\vec{r}_{t_n})\, d\vec{r}_{t_n} \tag{8}$$

where $\zeta - \zeta_n$ is assumed to be constant or independent of $\vec{r}_{t_n}$. Figure 1 illustrates what has been done to the surface in going from (6) to (7). In essence the continuous surface has been approximated by a non-overlapping collection of discrete flat areas which are all parallel to the z = 0 plane.

The derivative of the Green's function may be expressed in terms of a two dimensional inverse Fourier transform [5];

$$\frac{\partial g(\Delta\vec{r}_{t_n}, \Delta\zeta_n)}{\partial \Delta\zeta_n} = \frac{1}{2}\,\frac{4\pi}{(2\pi)^2} \int \mathrm{sgn}(\Delta\zeta_n) \exp\left[-j\sqrt{k_o^2 - |\vec{k}_\perp|^2}\,|\Delta\zeta_n| - j\vec{k}_\perp \cdot \Delta\vec{r}_{t_n}\right] d\vec{k}_\perp \tag{9}$$

where $\Delta\zeta_n = \zeta - \zeta_n$, $\Delta\vec{r}_{t_n} = \vec{r}_t - \vec{r}_{t_n}$ and the signum function is defined as follows;

$$\mathrm{sgn}(\Delta\zeta_n) = \begin{matrix} 1 & \Delta\zeta_n > 0 \\ 0 & \Delta\zeta_n = 0 \\ -1 & \Delta\zeta_n < 0 \end{matrix}$$

It is known that the average scattered field is specular [9,8] . This means that after averaging $J_q(\vec{r})$ over the stochastic height ζ , the resulting dependence of the averaged current on $\vec{r}_t$ is of the form $\exp(-j\vec{k}_{i_t} \cdot \vec{r}_t)$ where $\vec{k}_{i_t} = k_{ix}\hat{x} + k_{iy}\hat{y}$ is the transverse part of the incident wavenumber vector. In general, one cannot infer this particular functional dependence before averaging because ζ is a stochastic variable, i.e. $\zeta(\vec{r}_t)$. However, in the case of (8) where ζ and $\zeta_n, n = 1,2,\ldots,N$, are taken to be independent of $\vec{r}_t$ (because (8) represents a discrete approximation to the continuous surface) the following form is correct;

$$J_q(\vec{r}) = j(\zeta) \exp(-j\vec{k}_{i_t} \cdot \vec{r}_t) \tag{10}$$

Substituting (9) and (10) in (8) and performing the integration over $\vec{r}_{t_n}$ yields the following result;

$$j(\zeta)\exp(-jk_{i_t}\cdot\vec{r}_t) = J_{qo}^{\,i}\exp(-j\vec{k}_{i_t}\cdot\vec{r}_t - jk_{i_z}\zeta) - \sum_{n=1}^{N} j(\zeta_n)\,\mathrm{sgn}(\Delta\zeta_n)$$

$$\cdot\int \tilde{S}_n(\vec{k}_\perp - \vec{k}_{i_t})\exp\left\{-j\sqrt{k_o^2 - |\vec{k}_\perp|^2}\,|\Delta\zeta_n| - j\,\vec{k}_\perp\cdot\vec{r}_t + j(\vec{k}_\perp - \vec{k}_{i_t})\cdot\vec{r}_{t_{n_o}}\right\}d\vec{k}_\perp \tag{11}$$

where $J_{qo}^{\,i} = 2H_o\,\hat{q}\cdot[\hat{z}\times\hat{h}]$, $\tilde{S}_n(\cdot)$ is the Fourier transform of the $n^{\underline{th}}$ support, and $\vec{r}_{t_{n_o}}$ is the vector distance from the origin to the centroid of the $n^{\underline{th}}$ flat area projected on the $z=0$ plane. The curvatures and slopes of the continuous surface are taken to be so small that each of the discrete flat areas $A_n, n=1,2,\cdots,N$, have linear dimensions which are large in terms of a wavelength. Thus, the Fourier transforms of the support functions ($\tilde{S}_n$) will be very peaked about $\vec{k}_\perp = \vec{k}_{i_t}$ and they are approximated by delta functions. Accomplishing the $\vec{k}_\perp$-integration and dropping the common $\exp(-j\,\vec{k}_{i_t}\cdot\vec{r}_t)$ dependence yields

$$j(\zeta) \approx J_{qo}^{\,i}\exp(-j\,k_{iz}\,\zeta) - \sum_{n=1}^{N} j(\zeta_n)\,\mathrm{sgn}(\Delta\zeta_n)\exp(-j\,k_{iz}\,|\Delta\zeta_n|) \tag{12}$$

which is the equation that must be solved to find the current.

Before continuing on with the solution of (12), it is wise to review the essential rationale which permitted the simplification of (6) to (12). The first step entails the observation that the current at the point (x,y,ζ) may be effectively determined by its interaction with the current at an unspecified number of other points (x_n,y_n,ζ_n), $n=1,2,\cdots,N$, on the surface. In view of the very slowly undulating nature of the assumed surface, these N interacting points are approximated by large flat areas (see Figure 1). The elevations of these areas, namely ζ and $\zeta_n, n=1,2,\cdots,N$, are random but independent of the transverse spatial coordinates $\vec{r}_t$ and $\vec{r}_{t_n}$. As a result of this and the specular nature of the average scattered field, the dependence of the current upon the transverse coordinates $\vec{r}_t$ may be inferred directly as in (10). Finally, the height dependent part of the current is found to be a solution of (12). In essence, (12) relates the current on the area at an elevation ζ to the currents on the areas at elevations $\zeta_n, n=1,2,\cdots,N$.

As a first step toward solving (12) it should be noted that if there are no additional areas of interaction, i.e. $N=0$, or if the other areas are at the same elevation as the area at ζ, then

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) \tag{13}$$

which is the correct result for a single randomly elevated plane. Of particular significance here is the fact that (13) is independent of exactly how the limit of a single plane is achieved. That is, the proper limit is obtained either by allowing no additional areas ($N=0$) or by forcing all of the interacting areas to coalesce into one plane ($\zeta_n \to \zeta, n=1,2,\cdots,N$) . Equation (13) is also recognized to be the physical optics or single scatter approximation for the height dependent part of the current.

The most straightforward approach to solving (12) is to start with one additional area ($N=1$) and then build on this result. For $N=1$, (12) becomes

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) - j(\zeta_1) \operatorname{sgn}(\zeta - \zeta_1) \exp(-j k_{iz} |\zeta - \zeta_1|) \tag{14}$$

which relates the current on the area at $z=\zeta$ to the current induced by the incident field and the current on the area at $z=\zeta_1$. Rewriting (14) for the two regions $\zeta - \zeta_1 > 0$ and $\zeta - \zeta_1 < 0$ yields

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) - j(\zeta_1) \exp(-j k_{iz} [\zeta - \zeta_1]) \qquad \zeta - \zeta_1 > 0 \tag{14a}$$

and

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) + j(\zeta_1) \exp(j k_{iz} [\zeta - \zeta_1]) \qquad \zeta - \zeta_1 < 0 \tag{14b}$$

By inspection of these two equations, the following solution is obtained;

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) U(-\Delta\zeta_1) \tag{15}$$

where $\Delta\zeta_1 = \zeta - \zeta_1$ and $U(\cdot)$ is a unit step function defined as follows

$$U(-\Delta\zeta_1) = \begin{matrix} 0 & \Delta\zeta_1 > 0 \\ 1 & \Delta\zeta_1 \leq 0 \end{matrix} \tag{16}$$

(the fact that $U(-\Delta\zeta_1) = 1$ for $\Delta\zeta_1 = 0$ follows from the appearance of the signum function in (14) and its definition at $\Delta\zeta_1 = 0$). In verifying that (15) does indeed satisfy (14), one must be careful to reverse the roles of ζ and ζ_1 when substituting for $j(\zeta_1)$, i.e.

$$j(\zeta_1) = J_{qo}^{i} \exp(-j k_{iz} \zeta_1) U(\Delta\zeta_1)$$

Having found the solution of (12) for $N = 1$, it is a straightforward task to verify that adding more interacting areas simply results in multiplying (15) by more unit step functions. That is, the solution of (12) is as follows;

$$j(\zeta) = J_{qo}^{i} \exp(-j k_{iz} \zeta) \prod_{n=1}^{N} U(\zeta_n - \zeta) \tag{17}$$

This is an interesting and not altogether unexpected result. According to (17) only the lowest elevated area will support a nonzero current. For example, if the ζ-boundary is the lowest then $\Delta\zeta_n > 0$ for $n=1,2,\ldots,N$ and $j(\zeta)$ is just the current due to the incident field. Conversely, if ζ_i is the lowest then $j(\zeta_i)$ will equal the current due to the incident field and all of the other boundaries will support zero current. If there is no current on a boundary then the boundary produces no scattered field and so, for all intents and purposes, the boundary does not exist. Thus, (17) predicts that for any realization of the random boundaries, only one (the lowest) determines the scattering from the surface. This result is entirely reasonable because in discretizing the original integral equation, i.e. (6), the problem was recast into N + 1 interacting perfectly-conducting large areas, each having all the characteristics of a conducting half-space boundary. Having accomplished the mathematical solution correctly it is found that, for all intents and purposes, only one half-space boundary exists for any realization of the randomly elevated surface. Thus, the only consequence of (17) which does not have an immediate physical explanation is the fact that it is always the lowest boundary which gives rise to the scattered field. Unfortunately, no physical explanation for this particular result has been found.

The Average Scattered Field

To the degree of approximation provided by the Fraunhoffer diffraction integral, the far-zone scattered field in the direction $\vec{k}_s$ and at a distance R from the mean surface is given by

$$\vec{E}_s = j\frac{k_o \eta_o}{4\pi} g(R)\, \hat{k}_s \times \hat{k}_s \times \left\{ \sum_{m=0}^{M} \int_{A_m} \vec{J}(\vec{r}_{t_m}) \exp(j\vec{k}_{s_t}\cdot\vec{r}_{t_m} + jk_{sz}\zeta_m)\, d\vec{r}_{t_m} \right\} \qquad (18)$$

where $\vec{k}_s = \vec{k}_{s_t} + k_{sz}\hat{z} = k_o\hat{k}_s$, η_o is the impedence of free-space and g is given by (3). The sum is over all M + 1 flat areas comprising the total discretized surface and (ζ_o, A_o) represents the flat area having a height ζ and area A. Consistent with the previous stipulation that $A_m >> \lambda_o^2$, (18) reduces to the following

$$\vec{E}_s = j\,2\pi k_o \eta_o H_o\, g(R)\, [\hat{k}_s \times \hat{k}_s \times (\hat{z}\times\hat{h})]\, \delta(k_{sx}-k_{ix})\, \delta(k_{sy}-k_{iy})$$

$$\cdot \sum_{m=0}^{M} j(\zeta_m) \exp[j(k_{sz}-k_{iz})\zeta_m] \qquad (19)$$

If $\zeta_m = 0, m=0,1,2,\cdots$, this expression for $\vec{E}_s$ is too large by the factor M + 1; this error results from treating each A_m as essentially infinite and it can be rectified by dividing the rhs of (19) by $M+1$. Taking the average of the corrected version of (19) and noting that the average of $j(\zeta_m)\exp(jk_{sz}\zeta_m)$ is independent of m yields

$$<\vec{E}_s> = j\,2\pi\,k_o\,\eta_o\,H_o\,g(R)\,[\hat{k}_s\times\hat{k}_s\times(\hat{z}\times\hat{h})]\,\delta(k_{sx}-k_{ix})\,\delta(k_{sy}-k_{iy})$$

$$\cdot\left\langle j(\zeta)\,\exp[j(k_{sz}-k_{iz})\,\zeta\,]\right\rangle \tag{20}$$

where

$$j(\zeta) = \begin{cases} \exp(-jk_{iz}\zeta) & N = 0 \\ \exp(-jk_{iz}\zeta)\prod_{n=1}^{N} U(\zeta_n-\zeta) & N \geq 1 \end{cases} \tag{21}$$

The factor multiplying the <•> term in (20) is identical to the average scattered field, $\vec{E}_s^\circ$, for the case of a plane located at $z=0$ in the far-field approximation. Thus, $<\vec{E}_s>$ has the same direction as $\vec{E}_s^\circ$ and it differs in complex amplitude by the term <•> in (20). Since the scattered field is specular, $k_{sz} = \pm k_{iz}$ and for the field in the upper half-space (free-space) $k_{sz} = -k_{iz}$; thus, (20) becomes

$$<E_s>/E_s^\circ = \left\langle j(\zeta)\,\exp[-j\,2k_{iz}\,\zeta]\right\rangle \tag{22}$$

where $k_{iz} = -k_o\cos\theta$.

For $N=0$, (22) yields the following result;

$$<E_s/E_s^\circ> = \tilde{p}_1(2|k_{iz}|) \tag{23}$$

where $\tilde{p}_1$ is the characteristic function of ζ. For $N \geq 1$, there results

$$\langle E_s/E_s^\circ \rangle = \int_{-\infty}^{\infty} \exp(j\,2|k_{1z}|\,\zeta) \left\{ \int_{\zeta}^{\infty}\int_{\zeta}^{\infty} \cdots \int_{\zeta}^{\infty} p(\zeta_1,\zeta_2,\ldots,\zeta_N,\zeta)\,d\zeta_1 d\zeta_2 \cdots d\zeta_N \right\} d\zeta \tag{24}$$

where $p(\zeta_1,\zeta_2,\ldots,\zeta_N,\zeta)$ is the joint probability density function of the indicated random variables. This result for the relative strength of the average field involves two parameters - the degree of correlation between the non-overlapping areas and the number of areas which are important to the scattering. According to the similarity obtained in [8] and reviewed in the first section of this paper, total correlation represents the gently undulating surface ($\ell \gg \lambda_o$) while complete decorrelation now represents the uniformly rough surface ($\ell \ll \lambda_o$) limit. When the areas are all correlated, it can be shown using conditional densities that (24) reduces to (23) independent of N. This is as it should be because for $\ell \gg \lambda_o$ (24) represents the case of essentially a randomly elevated plane. When the non-overlapping areas are uncorrelated, (24) is very highly dependent upon N as should be expected since this represents the case of very strong mutual interaction on the surface. This, of course, is simply a reiteration of the fact that as the flat areas become decorrelated the mutual interaction effects on the surface increase. The remainder of this paper will be devoted to understanding and quantifying the importance of N for the case of uncorrelated surfaces ($\ell \ll \lambda_o$).

For an arbitrary joint density function representing uncorrelated heights, it is difficult to simplify (24). What is happening here is that the mathematics of probability theory are beginning to cloud the physics of the process. To overcome this limitation, consider the case where decorrelation implies statistical independence. For a statistically independent process, the joint density factors into a product of the marginal densities, i.e.

$$p(\zeta_1,\zeta_2,\ldots,\zeta_N,\zeta) = p(\zeta_1)p(\zeta_2)\cdots p(\zeta_N)p(\zeta)$$

The ζ_n-integration in (24) may be rewritten as follows;

$$\int_{\zeta}^{\infty} p(\zeta_n)d\zeta_n = 1 - \int_{-\infty}^{\zeta} p(\zeta_n)d\zeta_n = 1 - F(\zeta)$$

where $F(\zeta)$ is the common distribution function for $\zeta, \zeta_1, \zeta_2, \ldots, \zeta_N$; that is, all the random elevations are identically distributed because of the statistical homogeneity stipulation. Thus, when decorrelation implies statistical independence, (24) simplifies to the following form;

$$\langle E_s/E_s^{\circ}\rangle = \int_{-\infty}^{\infty} \exp(j\,2|k_{iz}|\,\zeta)p(\zeta)[1 - F(\zeta)]^N d\zeta \tag{25}$$

Since $F(-\infty) = 0$ and $F(+\infty) = 1$, the effect of the N interacting areas on the surface is to skew the integrand toward the negative range of ζ in (25). Unfortunately, it is difficult to proceed beyond this general result without a specific form for $p(\zeta)$ and $F(\zeta)$ for two reasons. First, it is difficult if not impossible to find a meaningful form for the integration in (25) for arbitrary $p(\zeta)$ and $F(\zeta)$. Second, and more importantly, is the simple fact that there is no formula for determining the effective number of interacting areas N; it must be determined by computing (25) as a function of N and <u>rationalizing the results with the physics of the rough surface scattering process</u>.

<u>Numerical Results and Interpretation</u>

For Gaussian surfaces (the only class of surface statistics for which the similitude has been proven [8]), the marginal density is

$$p(\zeta) = \frac{1}{\sqrt{2\pi\langle\zeta^2\rangle}} \exp(-\zeta^2/2\langle\zeta^2\rangle)$$

and

$$1 - F(\zeta) = \frac{1}{2}\,\mathrm{erfc}\left(\zeta/\sqrt{2<\zeta^2>}\right) \qquad (26)$$

where $<\zeta^2>$ is the mean square surface height and erfc(•) is the complementary error function. Substituting these forms in (25) and introducing the normalized variable $\eta = \zeta/\sqrt{2<\zeta^2>}$ yields;

$$<E_s/E_s^\circ> = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(j\, 2\sqrt{2}\ \Sigma\, \eta - \eta^2)[\mathrm{erfc}(\eta)/2]^N\, d\eta \qquad (27)$$

where $\Sigma = |k_{iz}|\sqrt{<\zeta^2>} = k_o\sqrt{<\zeta^2>}\cos\theta$ is the Rayleigh parameter. For arbitrary Σ , this integration is straightforward but must be accomplished numerically. For $\Sigma = 0$, the integral can be shown to yield $1/(N+1)$; consequently, as N increases (27) monotonically decreases. Figure 2 illustrates the behavior of the magnitude of the average scattered field as a function of Σ for $N = 0,1,2,$ and 3. Of particular note here is the fact that the curves begin to flatten out as both Σ and N increase. This figure also shows a very interesting result; for $N = 0$, 1, and 2 there is a specific range of Σ values for which each curve provides the maximum average scattered field compared to all the other curves (for $N = 0$ the range is $0 \leq \Sigma \lesssim 1$). Figure 3 shows a plot of N values as a function of the corresponding Σ range over which the average scattered field is maximized. That is, for each value of N and Σ corresponding to a point in Figure 3, equation (27) is maximized, i.e. any other N will provide a smaller value for (27).

In order to appreciate the implication of this result it is necessary to recall the meaning of N. As simulated in this analysis, N represents the effective number of uncorrelated areas on the surface which interact with the reference point on the surface. Equation (27) describes the influence of these N-area interactions on the coherent scattered field as a function of the

Rayleigh parameter or surface roughness. From a physical point of view, one should expect that for small roughness each point on the surface scatters independently of all the other points; hence, this corresponds to the $N = 0$ or no interaction. As the surface becomes rougher, it is certainly reasonable to expect that the region of interaction on the surface should increase; hence, N should also increase. With this bit of simple physics in mind, it is now possible to interpret the results in Figures 2 and 3. In essence, Figures 2 and 3 show the range of Σ values for which a given number of interacting areas on the surface are dominant or are most important to the scattering process. Figure 3 shows that for $\Sigma \lesssim 5$, N varies very nearly as Σ^3. Hence, the number of interacting points or regions on the surface is increasing as the cube of the roughness. To the author's knowledge, this is the first time that such a direct correspondence has been demonstrated.

It should be noted that the parameter N may be interpreted [11] somewhat differently than presented above. Although this secondary interpretation does not follow directly from the analysis presented here, it would certainly seem to apply to the original uniformly rough surface. In particular, the parameter N may be thought of as the number of additional reflections that an incident ray undergoes due to multiple scattering on the surface. From this point of view, each value of N is clearly only dominant over a limited range of roughness [11]. Furthermore, this analogy provides an alternate reasoning for moving from the $N = n_o$ to the $N = n_o + 1$ curve in Figure 2 as the Rayleigh parameter increases. That is, once the $N = n_o$ curve drops below the $N = n_o + 1$ curve in Figure 2, this signifies the need to include more multiple scattering or ray reflections on the surface. This interpretation and the results in Figure 3 imply that the order of multiple scattering or the number of ray reflections on the surface increases as the cube of Rayleigh parameter. Clearly, this interpretation is synonymous with the increasing area of interaction on the surface.

Given the above rationale for selecting N as a function of Σ, how then does one construct a continuous curve of the average scattered field as a function of Σ ? A technique that has been found to be relatively fast is the following. One chooses three contiguous values of N and computes (27) as a function of Σ for each of the three values. One then finds the value of Σ which maximizes (27) for the <u>middle</u> N and this provides one point on the plot of average scattered field strength versus Σ . This process is continued until a smooth curve can be drawn through the points. Because N varies as the third power of Σ , the maxima of (27) are very close together and so not many points are required before the trend of the maxima can be established. The only point where there will be some degree of interpolation required is near the transition from $N=0$ to $N=1$.

Figure 4 compares (27), as computed according to the above prescription, with the results of DeSanto and Shisha [6]. (Both results clearly show a larger average scattered power than is predicted by physical optics.) The disagreement is due most probably to the fact the integral equation for the average scattered field used by DeSanto and Shisha is only the first term in an infinite hierarchy of integral equations. The solution represented by (27) is also an approximation because the large flat areas comprising the substitute surface have essentially been replaced by planes. However, this simplification does not significantly alter the basic processes responsible for the effects of multiple interaction on the average scattered field and so it is reasonable to express a high degree of confidence in these results.

Conclusions

The primary purpose of this paper is to present an alternative approach to the problem of coherent scattering from a uniformly rough surfaces, i.e.

surfaces for which the correlation length is significantly smaller than the electromagnetic wavelength. The motivation for this effort is the need to obtain a degree of physical insight which is not present in the exact singular integral equation describing this problem. The approach is based upon the remarkable similarity [8], with respect to the intermediate mathematical results, between the stochastic Fourier transform of the total current on the gently undulating surface and the <u>part</u> of the stochastic Fourier transform of the current on the uniformly rough surface which is responsible for the average scattered field. This similarity leads to the <u>replacement</u> of the uniformly rough surface by a discretized version of the gently undulating surface comprising large, flat, non-overlapping, <u>uncorrelated</u> areas. This substitution is justified by the fact that the stochastic Fourier transform of the current on the discretized surface obeys exactly the same governing relationships as for the uniformly rough surface.

A second major step in the analysis entails identifying the effective number of interacting areas on the discretized surface in terms of orders of multiple scattering on the surface and recognizing the necessity to monotonically increase the order of multiple scattering with increasing surface roughness. In particular, it is found that the number of interacting areas on the surface, or alternatively the order of multiple scattering, necessary to maximize the average scattered field, varies as the cube of the Rayleigh roughness parameter. The maximum property of the average scattered field is consistent with earlier arguments [8] that the uniformly rough surface produces a maximum coherent field for a fixed Rayleigh parameter.

The essence of the approach is that of solving a simpler problem which can be shown to have the same mathematical properties with respect to the average scattered field as the actual surface. Clearly, it is desirable t compare the results obtained herein with a solution of the exact integral equation for the average scattered field. However, this must await further study of the integral equation and, in particular, the analytic properties of the average scattered field for "off-shell" ($k \neq -k_{iz}$) conditions.

Acknowledgements

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Figure Captions

Figure 1a. Geometry of the rough surface interface.

Figure 1b. Discrete approximation to the true rough surface in Figure 1a. The dashed continuous line is the original surface while the non-overlapping horizontal line segmets form the discrete approximation to the surface.

Figure 2. A plot of the coherent scattered power as a function of surface roughness for $N = 0, 1, 2, \& 3$. Note that for each N, there is a range of surface roughness for which the coherent power is larger than for all other values of N.

Figure 3. A plot of the N value which maximizes the average scattered power over the indicated range of surface roughness. The figure illustrates the increasing region of multiple interaction on the surface as the surface roughness increases. The solid line represents the relationship $N = \Sigma^3$.

Figure 4. A comparison of the coherent scattered power as a function of surface roughness as predicted by single scattering $(N = 0)$, DeSanto's & Shisha's computations, and equation (27). N, or the number of interacting points or regions on the surface, has been replaced by a continuous function of Σ which fits the results in Figure 3.

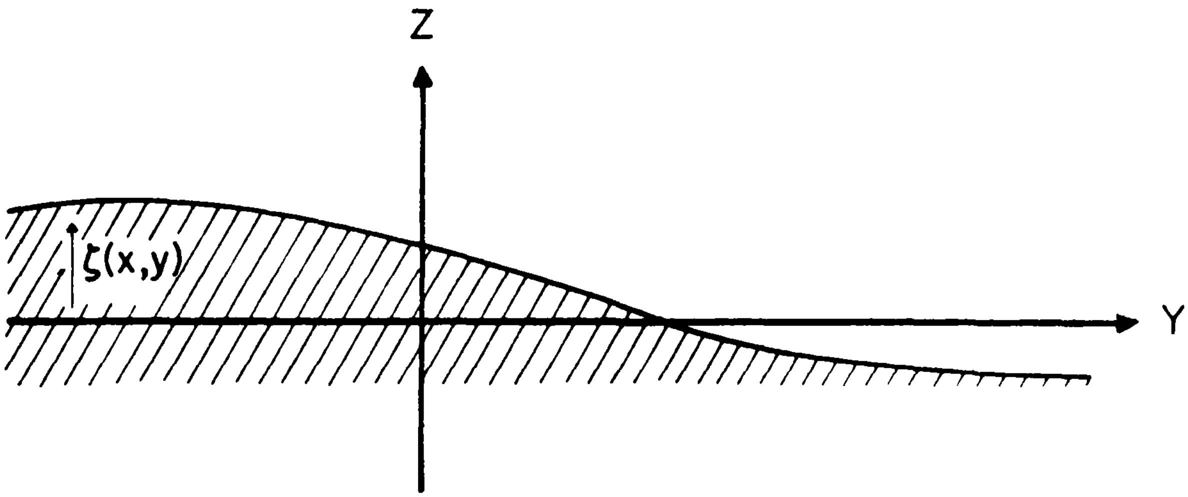


Figure 1a

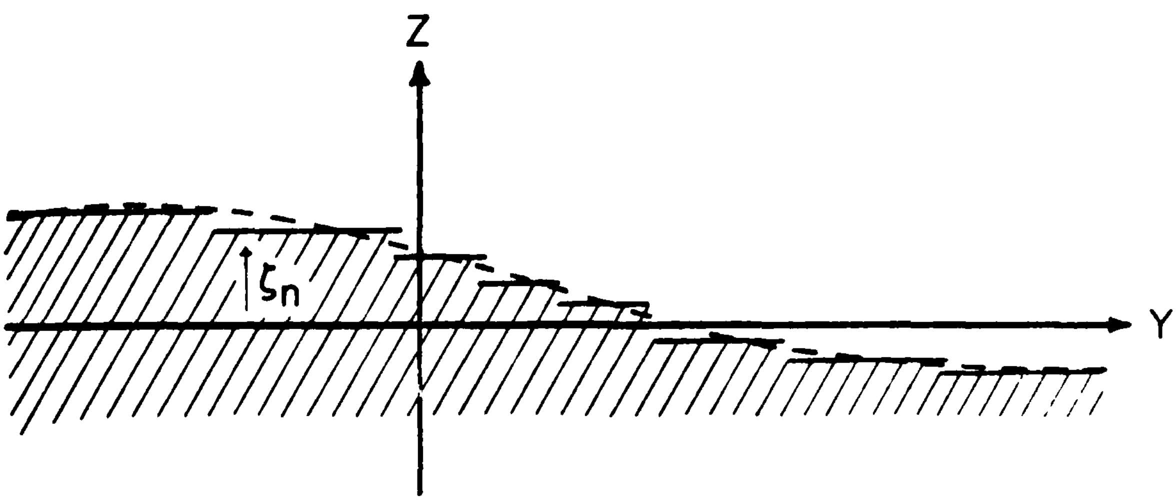


Figure 1b

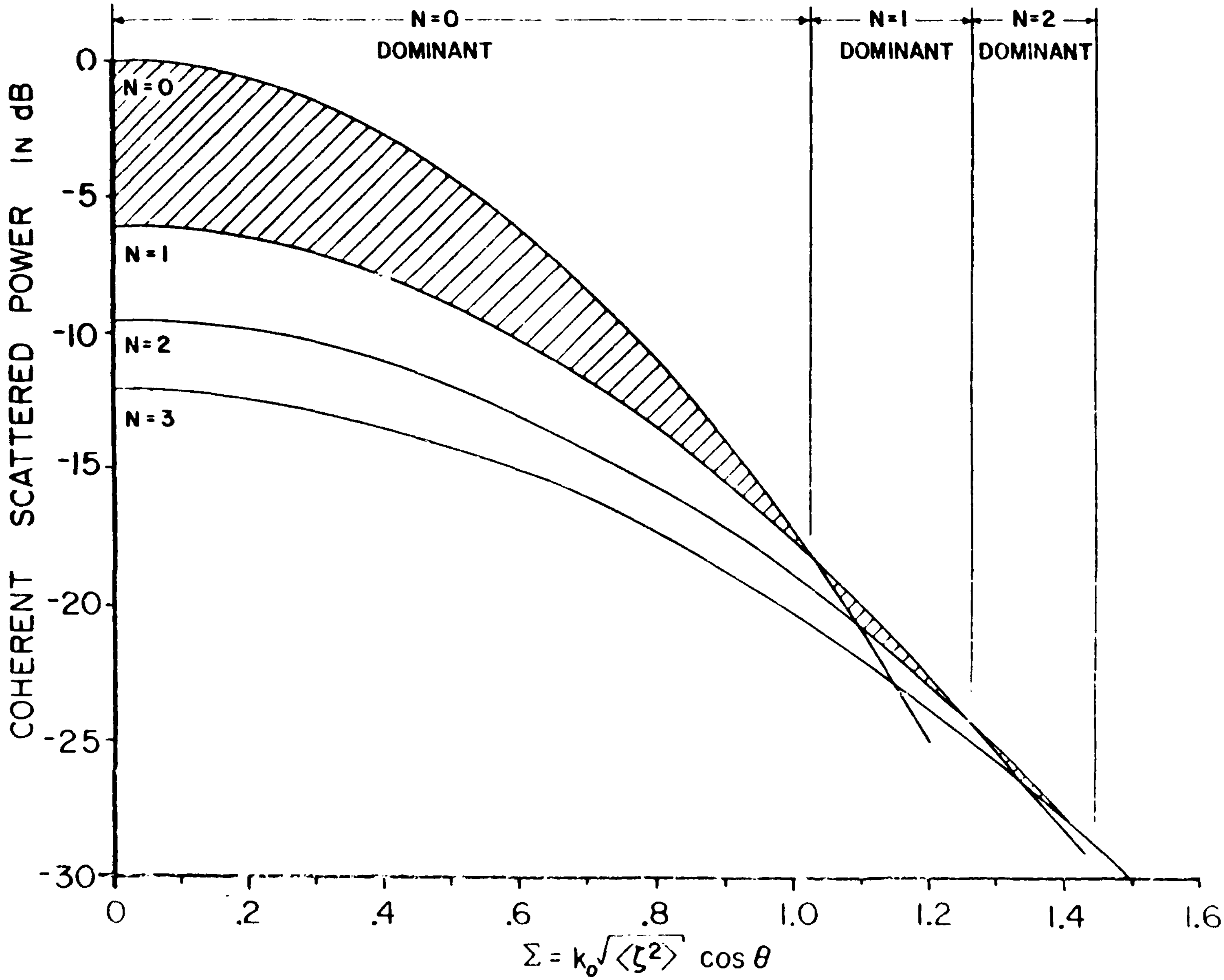
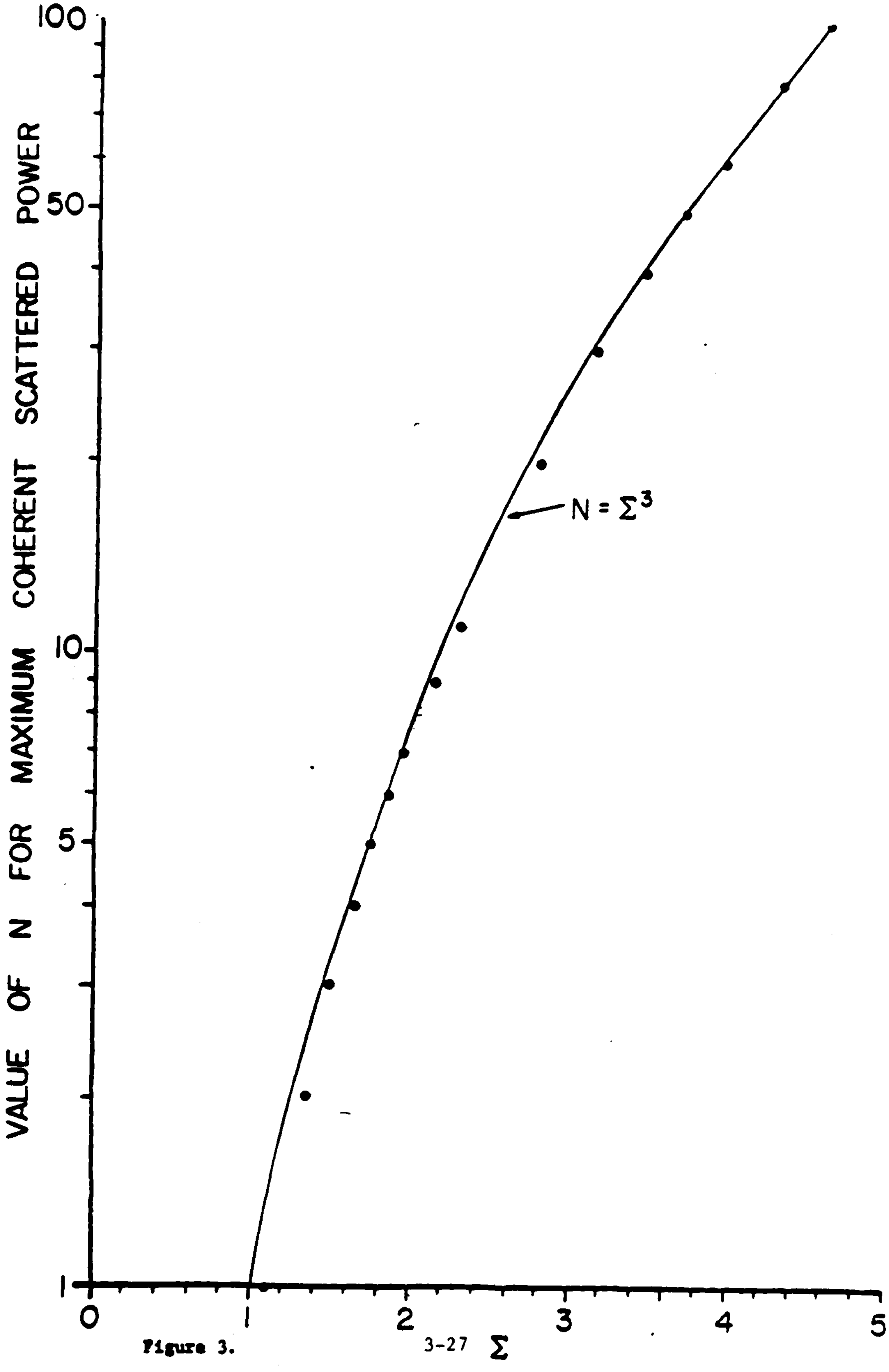

N=0
DOMINANT
N=1
DOMINANT
N=2
DOMINANT
N=0
N=1
N=2
N=3
COHERENT SCATTERED POWER IN dB
0
-5
-10
-15
-20
-25
-30
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
$\Sigma = k_o\sqrt{\langle\zeta^2\rangle}\cos\theta$

Figure 2.



100
50
10
5
1
0
1
2
3
4
5
VALUE OF N FOR MAXIMUM COHERENT SCATTERED POWER
N = Σ³
Σ

Figure 3.

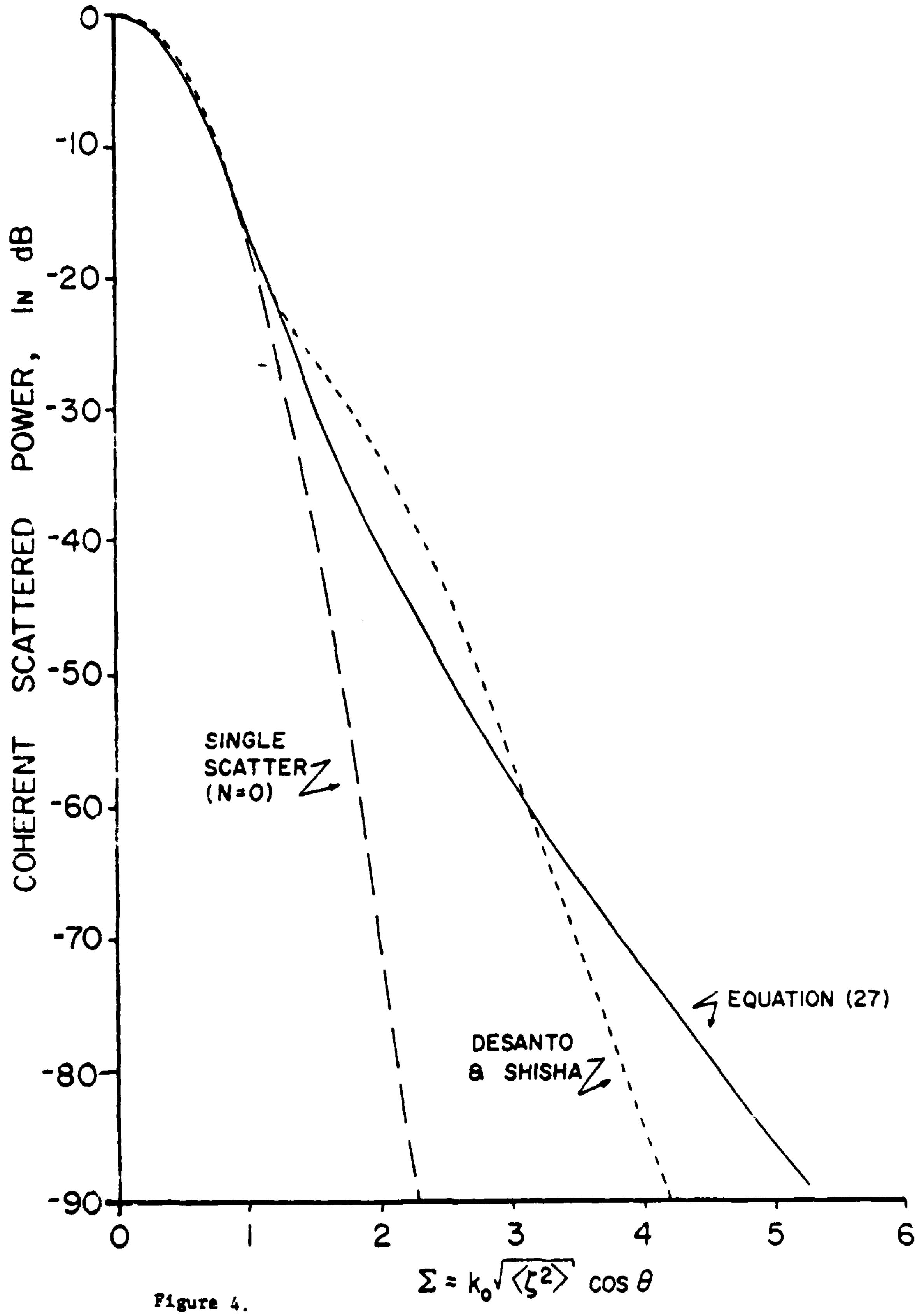

COHERENT SCATTERED POWER, IN dB
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
SINGLE SCATTER (N=0)
EQUATION (27)
DESANTO & SHISHA
0
1
2
3
4
5
6
$\Sigma = k_0\sqrt{\langle\zeta^2\rangle}\cos\theta$

Figure 4.

Scattering From A Class of Randomly Rough Surfaces

by

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Abstract

This paper develops new results pertaining to electromagnetic scattering from perfectly conducting random surfaces for which decorrelation does not imply statistical independence. Using an exact theory for the current induced on the surface and the far-field approximation for the scattered field, it is shown that the incoherent scattered power consists of two parts. The first part corresponds to the so-called diffuse scattered power. The second part is specular in its angular dependence and is a direct consequence of the fact that the two point joint density for the surface height, slopes, etc. does not reduce to the product of the single point joint densities for infinite separation distances. Computations for a gently undulating jointly exponentially distributed surface show that the incoherent specular power is equal to coherent scattered power for the Rayleigh parameter near one. When the Rayleigh parameter is large, the incoherent specular power is significantly larger than the coherent power. The analysis further indicates that scattering measurements provide an ideal way for identifying this class of surfaces.

Introduction and Summary

Frequently, when comparing measurements of scattering from randomly rough surfaces with theoretical models there are little or no data on the statistical properties of the surfaces. This limitation means that the surface statistics must be assumed. Unfortunately, the choice of the surface statistics is usually based upon analytical convenience rather than reality. If the assumed statistics are not a good approximation to the actual surface, this can lead to erroneous rejection of the theory or expensive additional experimentation.

Recently, Lennon and Papa [1980, 1981] employed digitized terrain maps to obtain estimates of very large scale surface height statistics which could then be used to predict scattering from the region in question [Papa, et al., 1980]. Because of the complexity of the theoretical scattering model, it is not practical to use measured statistics such as joint height histograms directly in the scattering computations. A reasonable alternative is to "best-fit" these statistics to a functional form for the joint height probability density function which, in turn, is amenable to use in the scattering theory. Since the functional forms for the joint height statistics may be quite general, scattering theories based upon Gaussian surface statistics are no longer adequate. Beckmann [1973] has considered the problem of scattering from non-Gaussian surfaces when the only height data available are the marginal density and the correlation function. However, his analyses and results are restricted to the case where decorrelation of the surface height implies statistical independence.

The purpose of this paper is to consider scattering from surfaces for which decorrelation does not imply statistical independence. The motivation for this study is threefold. First, there is no obvious reason why a real surface should be statistically independent when it is decorrelated. Also, from a mathematical point of view statistical independence is not, in general, implied by a lack of correlation [Papoulis, 1965]. Finally, there are a relatively large number of joint probability functional forms for which decorrelation and statistical independence are not synonymous [Lennon and Papa, 1981] and there is no apparent reason why these should be rejected _a priori_ when attempting to determine the "best fit" functional form.

The analysis starts with describing the far-zone scattered field by the standard far-field diffraction integral involving the current induced on the surface by the incident plane wave and the Fourier phase kernel. Averages necessary to compute the mean scattered field and its variance are converted to convolutions in a non-stochastic Fourier transform space [Brown, 1982]. The advantage of this approach is that the spatial dependence of the transformed current is known and this in turn, implies that the average scattered field is specular, e.g. it is nonzero for one and only one scattering angle. The second moment of the scattered field contains a specular component due to the nonconvergent nature of the two point joint density for the surface height, slopes, curvatures, etc. of the surface as the distance between the two points on the surface approaches infinity. When decorrelation implies statistical independence, this specular component is exactly canceled by the subtraction of the square of the mean scattered field; thus, the variance of the scattered field contains no specular components. However, if the surface is not statistically independent when it is uncorrelated, the _variance_ of the scattered field is shown to have a _nonzero specular component_ which is dependent upon the _difference_ between the decorrelated joint density and the square of the marginal or single point density for the surface height, slopes, curvatures, etc. This result is a simple consequence of the fact that no two points on the surface scatter _statistically_ independent of each other regardless of the distance between them.

In order to illustrate this effect, a very gently undulating surface which is amenable to the physical optics approximation for the induced surface current is assumed. The two point joint height probability density function is taken to be exponential and the specular part of the scattered field variance is compared to the square of the mean scattered field as a function of the Rayleigh roughness parameter (Σ). The specular part of the scattered field variance goes to zero as the Rayleigh approaches zero; however, as the Rayleigh parameter increases, it also increases to a peak value in the neighborhood of $\Sigma = 1$ where it is equal to square of the mean scattered field. For larger values of Σ, the specular part of the variance greatly exceeds the square of the mean scattered field; the former behaving asymptotically as Σ^{-3} while the latter goes as Σ^{-6}.

The importance of this analysis rests primarily in its implications. If the Rayleigh parameter is very small ($\Sigma \ll 1$) or very large ($\Sigma \gg 1$), it is doubtful that the specular component of the variance of the scattered field will be measurable. However, if the geometric and electromagnetic parameters of the experimental scattering system can be adjusted so that $\Sigma \sim 1$ then this specular part of scattered field variance should be detectable. If no specular part of the field variance is measured then one may safely conclude that the surface is statistically independent when it is decorrelated. A final implicacion of this analysis is that one should exercise great care in fitting arbitrary functional forms for joint probability densities to histogram data for purposes of predicting theoretical scattering from the surface.

Scattering Analysis

The stochastic surface $z = \zeta(x,y)$ is assumed to be a zero mean statistically homogeneous process. Beneath the surface, the medium is taken to be perfectly conducting while free-space comprises the medium above. A magnetic field of the form $\vec{H}^i(\vec{r}) = H_o \hat{h} \exp(-j\,\vec{k}_i \cdot \vec{r})$ is incident upon the surface; $\hat{h}$ is the polarization of the magnetic field, $\hat{k}_i = \vec{k}_i/k_o$ is its direction of propagation, and $k_o = 2\pi/\lambda_o$ is the wavenumber. Although the far-field scattering approximation leads to difficulties when dealing with unbounded illumination [Sancer, 1969], these problems can be circumvented by relating the results to flat surface "scattering" in the same approximation. With this caveat in mind, the far-zone scattered electric field is given by the following form;

$$\vec{E}_s = j \frac{k_o \eta_o}{4\pi} g(R_o) \int [\hat{k}_s \times \hat{k}_s \times \vec{J}(\vec{r})] \exp(j \vec{k}_s \cdot \vec{r}) \, d\vec{r}_t \tag{1}$$

where η_o is the impedance of free-space, $g(R_o) = \exp(-j k_o R_o)/R_o$, R_o is the distance from the origin of the surface based coordinated system to the far-field point, $\vec{k}_s/k_o = \hat{k}_s$ is the direction of the far-field point of observation, and $\vec{r}_t = x\vec{x} + y\vec{y}$ is a point on the $z=0$ plane. The quantity $\vec{J}(\vec{r})$ is defined as follows;

$$\vec{J}(\vec{r}) = \vec{J}_s(\vec{r})\left[1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2\right]^{1/2} \tag{2}$$

where $\vec{J}_s(\vec{r})$ is the current density on the surface and $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are the x and y slopes of the surface at the point $(\zeta,\vec{r}_t)$. Redefinition of the current as in (2) permits an integration over the $z=0$ plane in (1) rather than the actual surface. With $\Gamma = j k_o \eta_o g(R_o)/4\pi$, the average scattered field is given by

$$\langle\vec{E}_s\rangle = \Gamma \int \left[\hat{k}_s \times \hat{k}_s \times \langle\vec{J}(\vec{r}) \exp(j k_{sz} \zeta)\rangle\right] \exp(j \vec{k}_{s_t} \cdot \vec{r}_t) d\vec{r}_t \tag{3}$$

where $\vec{k}_{s_t} + k_{sz}\hat{z} = \vec{k}_s$ and the $\langle\cdot\rangle$ brackets denote an ensemble average over all stochastic quantities upon which $J(\vec{r})$ depends. With $p_1(\zeta,Z)$ equal to the single point joint probability density function for the stochastic variables ζ and Z, the average in (3) is as follows;

$$\langle \vec{J}(\vec{r}) \exp(jk_{sz}\zeta) \rangle = \iint \vec{J}(\vec{r}) \exp(jk_{sz}\zeta) p_1(\zeta, Z) d\zeta dZ \qquad (4)$$

where Z is symbolic of all the surface height derivatives which determine $\vec{J}(\vec{r})$. Since this number is in general infinite, Z stands for ∇ [illegible] $\zeta, \ldots$ [Brown, 1982] and the dependence of these higher order surface derivatives upon $\vec{r}_t$ is implied. With

$$\vec{J}(\vec{r}) = \hat{k}_s \times \hat{k}_s \times \vec{J}(\vec{r}) = (\hat{k}_s \cdot \vec{J})\hat{k}_s - \vec{J} \qquad (5)$$

and using (4), the squared magnitude of the average scattered field is easily shown to reduce to the following form;

$$|\langle \vec{E}_s \rangle|^2 = |\Gamma|^2 \iiiint\iint \left\{ \vec{J}(\vec{r}) \cdot \vec{J}^*(\vec{r}\,') \exp[jk_{sz}(\zeta - \zeta')] p_1(\zeta, Z) p_1(\zeta', Z') \right\}$$

$$\cdot \exp\left[j\vec{k}_{s_t} \cdot (\vec{r}_t - \vec{r}_t\,') \right] d\zeta d\zeta' dZ dZ' d\vec{r}_t d\vec{r}_t\,' \qquad (6)$$

where the symbol * denotes the complex conjugate operation and $p_1(\zeta', Z')$ is not conjugated because it is real. Following a similar development, the mean squared scattered field may be expressed as follows;

$$\langle |\vec{E}_s|^2 \rangle = |\Gamma|^2 \iiiint\iint \left\{ \vec{J}(\vec{r}) \cdot \vec{J}^*(\vec{r}\,') \exp[jk_{sz}(\zeta - \zeta')] p_2(\zeta, \zeta', Z, Z'; \Delta\vec{r}_t) \right\}$$

$$\cdot \exp\left[j\vec{k}_{s_t} \cdot (\vec{r}_t - \vec{r}_t\,') \right] d\zeta d\zeta' dZ dZ' d\vec{r}_t d\vec{r}_t\,' \qquad (7)$$

where $\Delta\vec{r}_t = \vec{r}_t - \vec{r}_t\,'$. The function p_2 in (7) is the two point joint probability density function which depends upon two sets of stochastic variables and the horizontal vector distance between them.

If the stochastic Fourier transforms are defined as follows;

$$\tilde{\vec{J}}(\vec{r}_t,k,K) = \iint \vec{J}(\vec{r})\exp[jk\zeta + jKZ]d\zeta dZ \tag{8a}$$

$$\tilde{p}_1(k,K) = \iint p_1(\zeta,Z)\exp[jk\zeta + jKZ]d\zeta dZ \tag{8b}$$

$$\tilde{\tilde{p}}_2(k,k',K,K';\Delta\vec{r}_t) = \iiiint p_2(\zeta,\zeta',Z,Z';\Delta\vec{r}_t)\exp[j(k\zeta + jk'\zeta' + KZ + K'Z')]d\zeta d\zeta' dZdZ' \tag{8c}$$

and it is noted that

$$F\left[\vec{J}^*(\vec{r}_t,\zeta,Z)\right] = \left\{\tilde{\vec{J}}(\vec{r}_t,-k,-K)\right\}^* \tag{8d}$$

where F symbolizes the stochastic Fourier transform operation, the integrations over the stochastic variables in (6) and (7) may be alternatively expressed as convolutions in transform space [Brown, 1982]. The advantage of this approach is that the new variables of integration, i.e. the transform coordinates, do not depend upon $\vec{r}_t$, $\vec{r}_t{}'$, or $\Delta\vec{r}_t$. Rewriting the integrals in (6) and (7) as noted above yields

$$|\langle\vec{E}_s\rangle|^2 = \frac{|\Gamma|^2}{(2\pi)^{2+2\overline{S_\infty}}}\iiiiiint\left\{\tilde{\vec{J}}(\vec{r}_t,k,K)\cdot\left[\tilde{\vec{J}}(\vec{r}_t',-k',-K')\right]^*\tilde{p}_1(k_{sz}-k,-K)\cdot\tilde{p}_1(-k_{sz}-k,-K)\right\}\exp\left[jk_{s_t}\cdot(\vec{r}_t-\vec{r}_t')\right]dkdk'dKdK'd\vec{r}_t d\vec{r}_t' \tag{9}$$

and

$$\langle|E_s|^2\rangle = \frac{|\Gamma|^2}{(2\pi)^{2+2\overline{S_\infty}}}\iiiiiint\left\{\tilde{\vec{J}}(\vec{r}_t,k,K)\cdot\left[\tilde{\vec{J}}(\vec{r}_t',-k',-K')\right]^*\tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';\Delta\vec{r}_t)\right\}\cdot\exp\left[j\vec{k}_{s_t}\cdot(\vec{r}_t-\vec{r}_t')\right]dkdk'dKdK'd\vec{r}_r d\vec{r}_t' \tag{10}$$

The quantity $\overline{S_\infty}$ is equal to $\sum_{i=2}^{\infty} i$ and it results from the dependence of $\vec{J}_s(\vec{r})$ on all orders of surface height derivatives [Brown, 1982]. The coherent scattered power is proportional to $|\langle\vec{E}_s\rangle|^2$ while the incoherent scattered power is proportional to $\mathrm{Var}(E_s) = \langle|\vec{E}_s|^2\rangle - |\langle\vec{E}_s\rangle|^2$ or the variance of the scattered field. Using (9) and (10), the variance of the scattered field may be written as follows;

$$\mathrm{Var}(E_s) = \frac{|\Gamma|^2}{(2\pi)^{2+2\overline{S_\infty}}} \iiiint\!\!\iint \tilde{\vec{J}}(\vec{r}_t,k,K) \cdot \left[\tilde{\vec{J}}(\vec{r}_t^{\,\prime},-k',-K')\right]^* \Big\{\tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';\Delta\vec{r}_t)$$

$$-\tilde{p}_1(k_{sz}-k,-K)\tilde{p}_1(-k_{sz}-k',-K')\Big\} dk\,dk'\,dK\,dK'\,d\vec{r}_t\,d\vec{r}_t^{\,\prime} \tag{11}$$

In view of (5) and (8a), the stochastic Fourier transform of $\vec{J}(\vec{r})$ is proportional to the stochastic Fourier transform of $\vec{J}(\vec{r})$. However, it has been shown [Brown, 1982] that, in order to satisfy the stochastic Fourier transformed magnetic field integral equation for the current, $\tilde{\vec{J}}(\vec{r}_t,k,K)$ must have the following dependence upon $\vec{r}_t$;

$$\tilde{\vec{J}}(\vec{r}_t,k,K) = \vec{j}(k,K)\exp(-j\vec{k}_{i_t}\cdot\vec{r}_t) \tag{12}$$

where $\vec{k}_{i_t}$ is defined by $\vec{k}_i = \vec{k}_{i_t} + k_{iz}\hat{z}$. That is, the stochastic Fourier transform of the slope normalized current exhibits the same dependence upon $\vec{r}_t$ as the incident magnetic field. Using (5), (8a), and (12) in (11) and converting to a $\Delta\vec{r}_t$ coordinate yields

$$\mathrm{Var}(E_s) = \frac{|\Gamma|^2 A}{(2\pi)^{2+2\overline{S_\infty}}} \iiint\!\!\iint \Big\{\sum_q (1-\hat{k}_s\cdot\hat{q})j_q(k,K)j_q^*(-k',-K')\Big\}$$

$$\cdot\Big\{\tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';\Delta\vec{r}_t) - \tilde{p}_1(k_{sz}-k,-K)\,\tilde{p}_1(-k_{sz}-k,-K)\Big\}$$

$$\cdot\exp\left[j\Delta\vec{r}_t\cdot(\vec{k}_{s_t}-\vec{k}_{i_t})\right] dk\,dk'\,dK\,dK'\,d\Delta\vec{r}_t \tag{13}$$

where A is the illuminated area and the summation is over the values $q = x, y$, and z.

It should be noted that (13) is an exact result involving no approximations other than the far field integral description of the scattering process. The function $\tilde{p}_1(\cdot,\cdot)$ is also called the single point joint characteristic function of the stochastic variables ζ and Z while $\tilde{\tilde{p}}_2(\cdot,\cdot,\cdot,\cdot;\Delta\vec{r}_t)$ is the two point joint characteristic function of ζ,ζ',Z, and Z'. The important point to note about (13) is that apart from the phase factor only $\tilde{\tilde{p}}_2$ depends upon $\Delta\vec{r}_t$. The explicit dependence of $\tilde{\tilde{p}}_2$ on $\Delta\vec{r}_t$ comes about through the surface height correlation function $R(\Delta\vec{r}_t)$ and its higher order derivatives where

$$R(\Delta\vec{r}_t) = \langle\zeta(\vec{r}_t{}' + \Delta\vec{r}_t)\zeta(\vec{r}_t{}')\rangle$$

When $|\Delta\vec{r}_t| \to \infty$, the correlation function and its higher order derivatives approach zero and this corresponds to the condition known as decorrelation. If decorrelation implies statistical independence then

$$\lim_{|\Delta\vec{r}_t|\to\infty} \tilde{\tilde{p}}_2 = \tilde{p}_1(k_{sz}-k,-K)\tilde{p}_1(-k_{sz}-k',-K') \tag{14}$$

or the two point characteristic function factors into a product of single point characteristic functions. When this occurs, the $\Delta\vec{r}_t$-integrand in (13) vanishes as $|\Delta\vec{r}_t| \to \infty$ and so it has a finite support. This implies that the $Var(E_s)$ will contain no delta function dependency upon $\vec{k}_{s_t} - \vec{k}_{i_t}$. Thus, when decorrelation implies statistical independence, $Var(E_s)$ will contain no specular terms because specularity in the far-field approximation implies a term which varies as $\delta(\vec{k}_{s_t} - \vec{k}_{i_t})$.

If decorrelation does not imply statistical independence then (14) is invalid and the term

$$\left\{ \lim_{|\Delta\vec{r}_t|\to\infty} \tilde{\tilde{p}}_2 - \tilde{p}_1\tilde{p}_1 \right\}$$

will be nonzero. The consequences of this result may be easily seen by regularizing the $\Delta\vec{r}_t$-integrand in (13) by subtracting and then adding terms containing the above limiting form. This operation leads to the following result;

$$\begin{aligned}
\mathrm{Var}(E_s) = {} & \frac{|\Gamma|^2 A}{(2\pi)^{2+2\overline{S_\infty}}} \iiiint\!\!\iint \left\{ \sum_q [1-\hat{k}_s\cdot\hat{q}]j_q(k,K)j_q^*(-k',-K') \right\} \\
& \cdot \left\{ \tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';\Delta\vec{r}_t) - \tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';|\Delta\vec{r}_t|=\infty) \right\} \\
& \cdot \exp\left[j\Delta\vec{r}_t\cdot(\vec{k}_{s_t}-\vec{k}_{i_t}) \right] dk\,dk'\,dK\,dK'\,d\Delta\vec{r}_t \\
& + \frac{|\Gamma|^2 A}{(2\pi)^{2\overline{S_\infty}}}\, \delta(\vec{k}_{s_t}-\vec{k}_{i_t}) \iiiint \left\{ \sum_q [1-\hat{k}_s\cdot\hat{q}]j_q(k,K)j_q^*(-k',-K') \right\} \\
& \cdot \left\{ \tilde{\tilde{p}}_2(k_{sz}-k,-k_{sz}-k',-K,-K';|\Delta\vec{r}_t|=\infty) - \tilde{p}_1(k_{sz}-k,-K)\tilde{p}_1(-k_{sz}-k',-K') \right\} dk\,dk'\,dK\,dK'
\end{aligned} \tag{15}$$

The second term in (15) is clearly specular in its angular dependence and it is entirely a consequence of the difference between decorrelation and statistical independence.

It might appear that this result is a freak consequence of the unbounded nature of the incident field; however, such is not the case. The analysis may be carried through using a plane wave having a finite support (a beam plane wave) provided that the linear dimensions of the illuminated area are large relative to the decorrelation length of the surface. The only essential difference between the limited area result and (15) will be the appearance of the diffraction pattern of the finite illuminated area. If the linear dimensions of the illuminated area are not large compared to the decorrelation length, the above results do not apply. However, this is an entirely different problem which will not be considered here.

Discussion

The appearance of a specular term in the expression for the incoherent power is certainly a bit surprising. Consequently, it is desirable, if not mandatory, to seek some physical explanation for this result. Unfortunately, no such explanation has been found primarily because the statistical moments of the scattered field are obtained by mathematical operations governed by the laws of probability rather than physics. From a very fundamental point of view, this problem is closely akin to inquiring into the physical significance of statistical dependence and this question, to the author's knowledge, has no satisfactory answer. When decorrelation implies statistical independence, there is no specular part of the incoherent power because the second moment of the scattered field behaves asymptotically (as $|\Delta\vec{r}_t|\to\infty$) exactly like the square of the mean scattered field. When decorrelation does not imply statistical independence, this is not the case because no two points on the surface, even as $|\Delta\vec{r}_t|\to\infty$, scatter statistically independent of each other. That is, the second moment of the scattered field does not approach the square of the mean field even as $|\Delta\vec{r}_t|\to\infty$. As a final word of caution, one should not confuse functional dependence with statistical dependence; they are, in general, completely unrelated mathematical concepts.

The appearance of the specular term in (15) coupled with the lack of any experimental scattering data showing this effect (to the author's limited knowledge) suggest that decorrelation does indeed imply statistical independence for real surfaces. However, the purpose of the present analysis is not to address this question but rather to consider the theoretical scattering implications of statistically dependent surfaces. It is therefore beneficial to illustrate the strength of the specular part of the incoherent scattered power using a relatively simple example. Such is the intent of the next section.

Numerical Example

In order to keep this example as simple as possible, the surface is stipulated to be very gently undulating. More specifically, the surface height spectrum is postulated to contain no spatial frequencies the order of or larger than the electromagnetic wavenumber k_o and the variances of all orders of surface height derivatives are taken to be arbitrarily small. Under these conditions the physical optics approximation for the current induced on the surface is valid, i.e.

$$\vec{J}_s(\vec{r}) = 2\hat{n}(\vec{r}) \times \vec{H}^i(\vec{r}) \tag{16}$$

where $\hat{n}(r) = [-(\partial\zeta/\partial x)\hat{x} - (\partial\zeta/\partial y)\hat{y} + \hat{z}]/[1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2]^{1/2}$ is the upward directed unit normal to the surface. Substituting (16) in (2), taking the stochastic Fourier transform of this result, and then separating the $\vec{r}_t$-dependence off as in (12) yields

$$j_q(k,K) = 2H_o(2\pi)^{S_\infty}\delta(k-k_{iz})\left[C_z^q\delta(K) - jC_x^q\delta'(K_{1x})\delta(K_{2y}) - jC_y^q\delta'(k_{1y})\delta(K_{2x})\right] \tag{17}$$

where $S_\infty = 1 + \overline{S_\infty}$ and the K's are transform variables associated with the following stochastic variables (with p = x or y);

$$K \leftrightarrow \nabla\zeta, \nabla^2\zeta, \cdots$$
$$K_{1p} \leftrightarrow \partial\zeta/\partial p$$
$$K_{2p} \leftrightarrow \partial\zeta/\partial p\ , \nabla^2\zeta\ , \nabla^3\zeta\ , \cdots$$

The constants $C_p^q = \hat{q}\cdot(\hat{p}\times\hat{h})$ with both p and q ranging independently over x, y, and z are related to the polarization of the incident field. Substituting

(17) in the second or specular term in (15), denoted as $\mathrm{Var}(E_s)_s$, yields the following result:

$$\mathrm{Var}(E_s)_s = |\Gamma|^2 (2\pi)^2 A\,\delta(\vec{k}_{s_t}-\vec{k}_{i_t})(2H_o)\left[(1-\hat{k}_s\cdot\hat{x})(C_z^{\,x})^2 + (1-\hat{k}_s\cdot\hat{y})(C_z^{\,y})^2\right]$$
$$\cdot\left\{\tilde{\tilde{p}}_2(k_{sz}-k_{iz},-k_{sz}+k_{iz};|\Delta\vec{r}_t|=\infty) - \tilde{p}_1(k_{sz}-k_{iz})\tilde{p}_1(-k_{sz}+k_{iz})\right\} \quad (18)$$

where $\tilde{\tilde{p}}_2$ and $\tilde{p}_1$ are now the joint two point and marginal characteristic functions, respectively, of the stochastic height <u>only</u>. Normalizing (18) by the squared magnitude of the field scattered by a flat surface, $|E_s^\circ|^2$, and noting that specularity implies $k_{sz} = -k_{iz}$ leads to the following;

$$\frac{\mathrm{Var}(E_s)_s}{|E_s^\circ|^2} = \tilde{\tilde{p}}(-2k_{iz},2k_{iz};|\Delta\vec{r}_t|=\infty) - \tilde{p}(-2k_{iz})\tilde{p}(2k_{iz}) \qquad (19)$$

The squared mean scattered field, similarly normalized, is given by

$$\frac{|\langle E_s\rangle|^2}{|E_s^\circ|^2} = \tilde{p}(-2k_{iz})\tilde{p}(2k_{iz}) \qquad (20)$$

The specular part of the incoherent power, as given in (19), should be compared to the first term in (15) in order to determine which is dominant. However, the first term in (15) will depend, in the physical optics approximation, primarily upon the slope statistics. Since (19) is governed by the height statistics, it is possible for these two terms to exist in almost any ratio. A more meaningful assessment of (19) results from comparing it to (20). Such a comparison, in essence, relates the specular portion of the incoherent power to the coherent power scattered from the surface.

For this example the joint probability density function is taken to be exponential [Lennon and Papa, 1981], e.g.

$$p(\zeta,\zeta';|\Delta\vec{r}_t|=\infty) = \frac{3}{2\pi\langle\zeta^2\rangle} \exp\left\{-\frac{\sqrt{\zeta^2+\zeta'^2}}{\sqrt{\langle\zeta^2\rangle/3}}\right\} \qquad (21)$$

where $\langle\zeta^2\rangle$ is the height variance. The joint characteristic function can, with the aid of Fourier transform tables [Campbell and Foster, 1961], be shown to have the following form;

$$\tilde{\tilde{p}}(k,k';|\Delta\vec{r}_t|=\infty) = \left[1 + \frac{\langle\zeta^2\rangle}{3}(k^2+k'^2)\right]^{-3/2} \qquad (22)$$

The marginal characteristic function may be obtained from (22) by taking $k'=0$. Substituting these results in (19) and (20) yields

$$\frac{\mathrm{Var}(E_s)_s}{|E_s^\circ|^2} = [1 + 8\Sigma^2/3]^{-3/2} - [1 + 4\Sigma^2/3]^{-3} \qquad (23)$$

and

$$\frac{|\langle E_s\rangle|^2}{|E_s^\circ|^2} = [1 + 4\Sigma^2/3]^{-3} \qquad (24)$$

where $\Sigma = k_o\sqrt{\langle\zeta^2\rangle}\cos\theta$ is the Rayleigh parameter and θ is the angle of incidence measured from the z-axis. Figure 1 illustrates the variation of $\mathrm{Var}(E_s)_s \big/ |E_s^\circ|^2$ (the incoherent specular power) and $|\langle E_s\rangle|^2 \big/ |E_s^\circ|^2$ (the coherent power) as a function of the Rayleigh parameter.

When the Rayleigh parameter is small, the incoherent specular power is considerably smaller than the coherent power as it should be. In the neighborhood of $\Sigma = 1$ the incoherent specular power is equal to the coherent power and for $\Sigma > 1$ it greatly exceeds the coherent power. For large Rayleigh parameter, the incoherent specular power behaves asymptotically like Σ^{-3} while the coherent power is proportional to Σ^{-6}.

It is obvious from the results in Figure 1 that the incoherent specular power cannot be ignored. Since the total power scattered from the surface must be in agreement with the incident power, the existence of the coherent specular power must be at the expense of the diffuse part of the incoherent power, i.e. the first term in (15). That is, the Rayleigh parameter may have to be relatively large before the diffuse part of the incoherent power is significant relative to the incoherent specular and the coherent powers. Thus, when comparing the diffuse power scattered from statistically independent and dependent surfaces (as $|\Delta\vec{r}_t| \to \infty$) with comparable height variances, one should expect to see a significant difference for small to moderate values of the Rayleigh parameter. For very large values of Σ this difference will disappear and the asymptotic theories of Barrick [1968] or Sancer [1969] will be valid. For statistically dependent surfaces, Figure 1 illustrates that the validity criterion for these asymptotic theories should be based upon a vanishingly small value of incoherent specular power rather than a negligible coherent power.

The reader is reminded that the purpose of this paper is to analytically describe the scattering from a surface for which decorrelation does not imply statistical independence. This paper does not advocate the existence of such surfaces in the real world; only measured statistics can answer this question. However, these results do show how electromagnetic scattering measurements can be used to identify such surfaces provided that great care is taken in separating the coherent from the incoherent scattered power. Finally, these analytical results also show that extreme caution should be exercised in fitting statistically dependent functional forms for the probability densities to limited histogram measurements if one is interested in conditions corresponding to $1 \lesssim \Sigma \lesssim 10$.

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Figure Captions

Figure 1. A comparison of the coherent and incoherent specular powers scattered by a perfectly conducting, exponentially distributed, gently undulating, random surface. Both powers are normalized by a common factor.

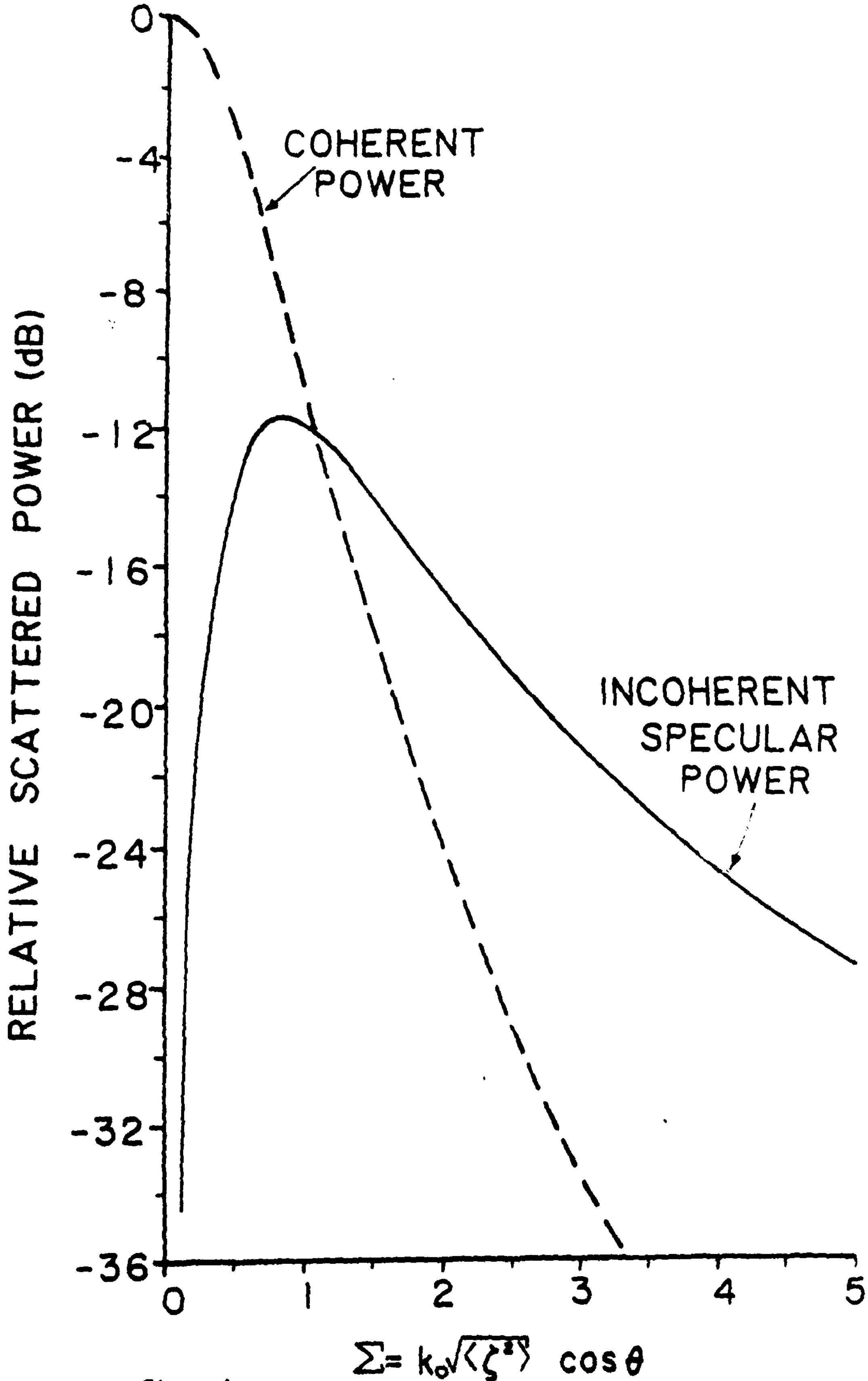
COHERENT POWER
INCOHERENT SPECULAR POWER
RELATIVE SCATTERED POWER (dB)
0
-4
-8
-12
-16
-20
-24
-28
-32
-36
0
1
2
3
4
5
$\Sigma = k_o\sqrt{\langle\zeta^2\rangle}\ \cos\theta$

Figure 1.

Scattering From Randomly Rough Surfaces And The Far Field Approximation

by

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Abstract

This paper derives rigorous results pertaining to the validity of the far field approximation for scattering from randomly rough, perfectly conducting surfaces having arbitrary statistics. The methodology employs the stochastic Fourier transform of the current induced on the infinite surface by either a bounded or unbounded incident plane wave. The results are general in that no approximate simplifying forms for the current are employed. Exact expressions are obtained for the mean and variance of the scattered field for unbounded illumination and they are compared to the far field approximations to illustrate how the latter simplifications fail in this limit. Some of the pitfalls of the far field approximation in the case of beam illumination are discussed. When the incident plane wave is bounded, the conventional far field form for the mean scattered field can be rigorously derived for arbitrary surfaces provided the cross sectional area of the incident beam is large compared to the square of the electromagnetic wavelength. The conventional far field result for the variance of the scattered field is shown to require the additional stipulation that the cross sectional area of the incident beam contains many decorrelation intervals of the surface roughness. The results obtained herein are important because they hold for arbitrary surface statistics. Whereas they appear to duplicate previous results, it must be remembered that the earlier results were only valid for a special class of surface statistics, i.e. surfaces for which single scattering theory holds.

Introduction and Summary

One of the most useful descriptions of a deterministic body scattering (or radiating) electromagnetic energy is the so-called far field approximation. The general formula for the electric field scattered by a perfectly conducting body and evaluated at a point $\vec{R}_o$ exterior to the body is as follows;

$$\vec{E}_s(\vec{R}_o) = -j \frac{\eta_o}{k_o} \nabla_o \times \nabla_o \times \int \vec{J}_s(\vec{r}) G(|\vec{R}_o - \vec{r}|) \, dS \qquad (1)$$

where $\eta_o = \sqrt{\mu_o/\varepsilon_o}$ is the characteristic impedance of free space, $k_o = 2\pi/\lambda_o$ is the electromagnetic wavenumber, G is free space Green's function $\exp(-j k_o |\vec{R}_o - \vec{r}|)/4\pi|\vec{R}_o - \vec{r}|$, and $\vec{J}_s$ is the current density on the surface S of the body. If the point $\vec{R}_o$ is sufficiently far removed from the body that (2) may be approximated by the following expression [Van Bladel, 1964]

$$\vec{E}_s(\vec{R}_o) \simeq j k_o \eta_o G(R_o) \hat{k}_s \times \hat{k}_s \times \int \vec{J}_s(\vec{r}) \exp(j k_o \hat{k}_s \cdot \vec{r}) \, dS \qquad (2)$$

where $\hat{k}_s$ is a unit vector specifying the direction of the point $\vec{R}_o$, then (2) is essentially the definition of the far field scattered by the body. The results of extensive computations by Hansen and Bailin [1959] have provided much insight into the magnitude of $\vec{R}_o$ necessary to accurately replace (1) by (2) for deterministic currents. Consequently, the range of validity of the far field approximation is reasonably well understood for deterministic problems.

When the geometry of the scattering body varies on a sample-by-sample basis so that it is more conveniently described in a probabilistic manner, the concept and implications of a far field are less well understood. The lack of clarity for this situation stems from two difficulties which are not encountered in the deterministic case. First, for the deterministic geometry one may postulate certain cannonical current distributions, compute $\vec{E}_s$ from (1) and (2), and obtain very useful information on the range of validity of the far field approximation. However, for the random geometry this cannot be done because the dependency of the current upon the random geometry is critical and it must be obtained (not postulated) from the boundary equations. The second difficulty stems from the need, for the random geometry problem, to know not only $\vec{E}_s$ but all or a portion of its statistical moments and these are not as simply related as the scattered field and power in the deterministic case.

A few studies have obtained useful results on the random geometry-far field problem [Barrick, 1965], [Miller, 1982], [Fung and Eom, 1982]; however, they are limited by the use of an approximate form for the current and only the inclusion of a quadratic phase term in (2). What is really needed is some general insight into the far-field approximation, as it applies to the random geometry situation, which is not limited by restrictive simplifications. Such is the goal of this paper.

The first problem to be considered is that of a randomly rough, perfectly conducting surface which is infinite in horizontal extent and illuminated by an unbounded plane wave. This infinite geometry/illumination case is selected as the starting point because the current on the surface assumes a somewhat simplified form and this permits the derivation of some useful exact results. Although the far field assumption is not applicable to this problem, results obtained from this approximation are reviewed in order to illustrate the functional dependencies predicted by it. The mean and the variance of scattered field are next obtained using the exact expression for the scattered field, i.e. (1). The mean scattered field is shown to be a plane wave propagating away from the surface in the specular direction. The amplitude of this plane wave is reduced relative to the amplitude of the incident plane wave by the effects of the roughness on the surface. The polarization of the scattered plane wave is dependent upon the multiple scattering processes occurring on the surface and this shows that the mean scattered field, contrary to previous statements [Moore and Williams, 1957], is governed by the laws of diffraction and not reflection.

The exact expression for the variance of the scattered field is found to be considerably more complicated than the mean scattered field relationship. However, the exact expression clearly shows that the far field approximation is not applicable when the surface and the illumination are unbounded even though the point of observation is taken to infinity.

The results obtained for the unbounded geometry/illumination problem are important from a theoretical point of view but they have little significance relative to the practical situation wherein the illumination is bounded, i.e. a beam plane wave. When the incident illumination is bounded, the problem becomes difficult because the surface current is more complicated. However, it is shown that when the illuminated area is very large compared to λ_o^2, the current appropriate to the unbounded illumination case in conjunction with the bound on the incident illumination may be used with little error.

This result for the current is then used in (1) for the purpose of computing the mean and variance of the scattered field. When multiple scattering on the surface is negligible, this procedure produces an exact result. When multiple scattering is important, the procedure leads to an underestimation of the true illuminated area on the surface, i.e. the true illuminated area is the area over which the current is nonzero. However, as long as the geometric cross sectional area of the incident beam is large compared to λ_o^2, the error in this procedure is small. Proceeding with this approach, it is possible to show in a very clear manner why the far field approximation for the scattered field and the case of a randomly rough scattering surface is not valid as the illuminated area becomes infinite. Finally, conditions are derived under which the far field simplification is valid for arbitrary surfaces.

While the far field approximation has been used extensively in randomly rough surface scattering problems, justification of the approximation has heretofore been limited at best. The results presented in this paper provide a general foundation for the approximation and yield insight into why it fails for an infinite surface.

Background

The surface described by $z = \zeta(x,y)$ is assumed to be infinite in the x and y directions and for $z > \zeta(x,y)$ the medium is taken to be free space while for $z < \zeta(x,y)$ the medium is perfectly conducting. The stochastic surface height $\zeta(x,y)$ comprises a zero mean, statistically homogeneous process with the mean surface corresponding to the $\cdot = 0$ plane. A plane wave illuminates the surface and its spatial support in the x-y plane will be taken to be either bounded, i.e. a beam plane wave, or infinite. The explicit form of the incident plane wave is as follows;

$$\vec{E}_i = E_o \exp(-j\,\vec{k}_i\cdot\vec{r})\,\Theta(\vec{r}_t)\,\hat{e}_i \qquad (3)$$

where $\vec{k}_i = k_o\hat{k}_i = \vec{k}_{i_t} + k_{i_z}\hat{z}$ and $\hat{k}_i$ specifies its direction of propagation. The function $\Theta(\vec{r}_t)$ represents the support of the incident plane and its dependence upon $\vec{r}_t = x\hat{x} + y\hat{y}$ signifies that it is independent of the z-dimension.

The mean field scattered in the direction $\hat{k}_s$ and having a polarization $\hat{e}$ is, in the far field approximation, given by the following;

$$<\vec{E}_s \cdot \hat{e}> = \Gamma \hat{e} \cdot \hat{k}_s \times \hat{k}_s \times \int \left\langle \vec{J}_s(\vec{r}) \exp(j k_{s_z} \zeta + j \vec{k}_{s_t} \cdot \vec{r}_t) dS \right\rangle$$

where $\vec{k}_{s_t} + k_{s_z} \hat{z} = k_o \hat{k}_s$ and $\Gamma = j k_o \eta_o G(R_o)$. To simplify matters somewhat, the integration over the rough surface is transformed into an integration over the z = 0 plane through $dS = \sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}\, d\vec{r}_t$ where $d\vec{r}_t = dxdy$ and $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are the surface slopes in the x and y-directions. Thus, defining the modified current $\vec{J}(\vec{r})$ as

$$\vec{J}(\vec{r}) = \sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}\; \vec{J}_s(\vec{r}) \quad ,$$

the far field approximation for $<\vec{E}_s \cdot \hat{e}>$ becomes

$$<\vec{E}_s \cdot \hat{e}> = \Gamma \hat{e} \cdot \hat{k}_s \times \hat{k}_s \times \int \left\langle \vec{J}(\vec{r}) \exp(j k_{s_z} \zeta) \right\rangle \exp(j \vec{k}_{s_t} \cdot \vec{r}_t)\, d\vec{r}_t \tag{4}$$

The average in (4) may be explicitly written as follows;

$$\left\langle \vec{J}(\vec{r}) \exp(j k_{s_z} \zeta) \right\rangle = \int \cdots \int \vec{J}(\vec{r}) p_1(\zeta, \nabla\zeta, \nabla^2\zeta, \cdots) \exp(j k_{s_z} \zeta)\, d\zeta d\nabla\zeta \cdots \tag{5}$$

where $\nabla\zeta$ is symbolic for the surface slopes, $\nabla^2\zeta$ is symbolic for the surface curvatures, etc. The function p_1 in (5) is the single point joint probability density function for ζ, $\nabla\zeta$, $\nabla^2\zeta$, etc. The integrations in (5) may also be written as convolutions of transforms with respect to the stochastic variables, i.e.

$$<\cdot> = \frac{1}{(2\pi)^{s_\infty}} \int \cdots \int \tilde{\vec{J}}(\vec{r}_t, k_1, \vec{k}_2, \cdots)\, \Phi_1(k_{s_z} - k_1, -\vec{k}_2, \cdots)\, dk_1 d\vec{k}_2 \cdots \tag{6}$$

where $s_\infty = \sum_{i=1}^{\infty} i$ and

$$\tilde{\vec{J}} = \int \cdots \int \vec{J}(\vec{r}) \exp\left(jk_1\zeta + j \sum_{n=1}^{\infty} \nabla^n \zeta \cdot \vec{k}_{n+1}\right) d\zeta \, d\nabla\zeta \cdots \qquad (7a)$$

$$\Phi_1 = \int \cdots \int p_1(\zeta, \nabla\zeta, \cdots) \exp\left(jk_1\zeta + j \sum_{n=1}^{\infty} \nabla^n \zeta \cdot \vec{k}_{n+1}\right) d\zeta \, d\nabla\zeta \cdots \qquad (7b)$$

With $\vec{K}$ signifying the dependence upon $\vec{k}_2, \vec{k}_3, \cdots$, a substitution of (6) into (4) yields the following;

$$\langle \vec{E}_s \cdot \hat{e} \rangle = \Gamma \, \hat{e} \cdot \hat{k}_s \times \hat{k}_s \times \frac{1}{(2\pi)^{s_\infty}} \iiint \tilde{\vec{J}}(\vec{r}_t, k_1, \vec{K}) \, \Phi_1(k_{s_z} - k_1, -\vec{K}) \exp(j\vec{k}_{s_t} \cdot \vec{r}_t) \, d\vec{r}_t \, dk_1 \, d\vec{K} \qquad (8)$$

It should be noted that Φ_1 is the single point joint characteristic function for the random variables $\zeta, \nabla\zeta, \nabla^2\zeta, \cdots$.

In order to proceed further one must have some knowledge of the stochastic Fourier transform of the modified current, i.e. $\tilde{\vec{J}}(\vec{r}_t, k_1, \vec{K})$. In particular, the variation of $\tilde{\vec{J}}$ with $\vec{r}_t$ is essential to determining the directional characteristics of the mean scattered field, i.e. the dependence of $\langle \vec{E}_s \cdot \hat{e} \rangle$ upon $\vec{k}_{s_t}$. The only case for which the <u>exact</u> $\vec{r}_t$-dependence of $\tilde{\vec{J}}$ is known is when the incident plane wave is unbounded or $\Theta(\vec{r}_t) = 1$ for all $\vec{r}_t$. When this occurs, it has been shown [Brown, 1982b] that $\tilde{\vec{J}}$ assumes the following form;

$$\tilde{\vec{J}}(\vec{r}_t, k_1, \vec{K}) = \vec{j}(k_1, \vec{K}) \exp(-j\vec{k}_{i_t} \cdot \vec{r}_t) \qquad (9)$$

Substituting (9) in (8) and taking $\hat{e}$ orthogonal to $\hat{k}_s$ in the specular direction yields

$$\langle \vec{E}_s \cdot \hat{e} \rangle = (2\pi)^2 \, \Gamma \, \delta(\vec{k}_{s_t} - \vec{k}_{i_t}) \iint \hat{e} \cdot \vec{j}(k_1, \vec{K}) \, \Phi_1(k_{s_z} - k_1, -\vec{K}) \, dk_1 d\vec{K} / (2\pi)^{s_\infty} \qquad (10)$$

This result indicates that the mean scattered field is a spherical wave modified

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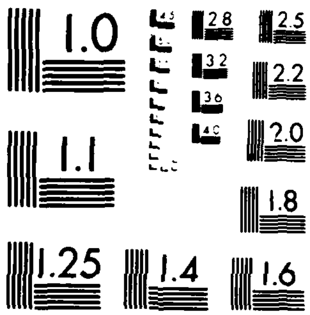

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by a product of two amplitude weighting factors - the δ-function and the integral term. The integral term is entirely a consequence of the surface roughness. The $\delta(\vec{k}_{s_t}-\vec{k}_{i_t})$ factor, loosely interpreted, shows that the scattered field is nonzero only in the direction $\hat{k}_{s_t}=\hat{k}_{i_t}$. Furthermore, since the point of observation is above the surface the condition $k_{s_z} = -k_{i_z}$ must also be satisfied. The form of (10) is certainly less than satisfying and, in fact, only has meaning when compared to the field "scattered" by a flat plane in the far field approximation [Brown, 1982a]. In short, (10) cannot be taken literally but, instead, must be interpreted.

The second moment of $\vec{E}_s \cdot \hat{e}$, in the far field approximation, is obtained by forming the product of (2) and its complex conjugate and then averaging over all stochastic variables. The averaging operation requires use of the two point joint probability density function, i.e. $p_2(\zeta_1,\zeta_2,\nabla\zeta_1,\nabla\zeta_2,\cdots;\vec{r}_{t_1}-\vec{r}_{t_2})$, which depends not only upon the random surface characteristics at $\vec{r}_{t_1}$ and $\vec{r}_{t_2}$ but also on the vector distance $\vec{r}_{t_1}-\vec{r}_{t_2}$. Since $\vec{E}_s$ is polarized orthogonal to $\hat{k}_s$ in the far field approximation, $\hat{e}$ is also chosen to be orthogonal to $\hat{k}_s$ and the second moment, using the far field approximation, can be written as follows;

$$<|\vec{E}_s\cdot\hat{e}|^2> = |\Gamma|^2\iint[\vec{J}(\vec{r}_1)\cdot\hat{e}][\vec{J}^*(\vec{r}_2)\cdot\hat{e}]p_2(\zeta_1,\zeta_2,\nabla\zeta_1,\nabla\zeta_2,\cdots;\vec{r}_{t_1}-\vec{r}_{t_2})\exp[j\vec{k}_s\cdot(\vec{r}_1-\vec{r}_2)]$$

$$\cdot\, d\vec{r}_{t_1}d\vec{r}_{t_2}d\zeta_1 d\zeta_2 d\nabla\zeta_1 d\nabla\zeta_2\cdots$$

Converting the integrations over the stochastic variables to convolutions in transform space and using (9), the above result can be manipulated into the following form;

$$<|E_s \cdot e|^2> = \lim_{A\to\infty} \frac{A|\Gamma|^2}{(2\pi)^{2s_\infty}} \int\cdots\int [\vec{j}(k_1,\vec{K}_1)\cdot\hat{e}][\vec{j}^*(-k_2,-\vec{K}_2)\cdot\hat{e}]$$

$$\cdot\, \Phi_2(k_{s_z}-k_1,-k_{s_z}-k_2,-\vec{K}_1,-\vec{K}_2;\Delta\vec{r}_t)\exp\left[j(\vec{k}_{s_t}-\vec{k}_{i_t})\cdot\Delta\vec{r}_t\right] d\Delta\vec{r}_t dk_1 dk_2 d\vec{K}_1 d\vec{K}_2 \quad (11)$$

where A is the illuminated area and

$$\Phi_2(k_1,k_2,\vec{K}_1,\vec{K}_2;\Delta\vec{r}_t) = \int\cdots\int p_2(\zeta_1,\zeta_2,\partial\zeta_1,\partial\zeta_2;\Delta\vec{r}_t)\exp\Big[j(k_1\zeta_1 + k_2\zeta_2$$

$$+ \vec{K}_1\cdot\partial\zeta_1 + \vec{K}_2\cdot\partial\zeta_2)\Big]\, d\zeta_1 d\zeta_2 d(\partial\zeta_1)d(\partial\zeta_2) \quad (12)$$

is the two point joint characteristic function with $\partial\zeta_1$ and $\partial\zeta_2$ symbolic for all orders of surface height derivatives $\nabla^n\zeta, n-1,2,\cdots$. Assuming that the surface statistics are such that decorrelation implies statistical independence yields

$$\lim_{|\Delta\vec{r}_t|\to\infty} p_2(\zeta_1,\zeta_2,\partial\zeta_1,\partial\zeta_2;\Delta\vec{r}_t) = p_1(\zeta_1,\partial\zeta_1)p_1(\zeta_1,\partial\zeta_2) \quad (13)$$

This condition implies that Φ_2 will have infinite support with respect to $\Delta\vec{r}_t$ and so the transform of Φ_2 contained in (11) will yield a δ-function dependence upon $\vec{k}_{s_t}-\vec{k}_{i_t}$. This problem can be overcome by using a function in the integrand of (11) which has a finite support with respect to $\Delta\vec{r}_t$. Thus, by subtracting and adding $\Phi_2(\cdot,\cdot;\infty)$ to the integrand of (11) yields

$$<|\vec{E}_s\cdot\hat{e}|^2> = \lim_{A\to\infty} \frac{A|\Gamma|^2}{(2\pi)^{2s_\infty}} \int\cdots\int [\vec{j}(k_1,\vec{K}_1)\cdot\hat{e}][\vec{j}^*(-k_2,-\vec{K}_2)\cdot\hat{e}]\Big\{\Phi_2(k_{s_z}-k_1,-k_{s_z}-k_2,-\vec{K}_1,-\vec{K}_2;\Delta\vec{r}_t)$$

$$-\, \Phi_2(k_{s_z}-k_1,-k_{s_z}-k_2,-\vec{K}_1,-\vec{K}_2;\infty)\Big\} \exp\left[j(\vec{k}_{s_t}-\vec{k}_{i_t})\cdot\Delta\vec{r}_t\right] d\Delta\vec{r}_t dk_1 dk_2 d\vec{K}_1 d\vec{K}_2$$

$$+ \lim_{A\to\infty} \frac{A|\Gamma|^2}{(2\pi)^{2(s_\infty-1)}}\, \delta(\vec{k}_{s_t}-\vec{k}_{i_t}) \int\cdots\int [\vec{j}(k_1,\vec{K}_1)\cdot\hat{e}][\vec{j}^*(-k_2,-\vec{K}_2)\cdot\hat{e}]$$

$$\Phi_2(k_{s_z}-k_1,-k_{s_z}-k_2,-\vec{K}_1,-\vec{K}_2;\infty)\, dk_1 dk_2 d\vec{K}_1 d\vec{K}_2 \quad (14)$$

Because of (13), $\Phi_2(\cdot,\cdot;\infty)$ may be rewritten as a product of marginal characteristic functions. Furthermore, since the surface roughness is a zero mean process, the marginal characteristic functions will be real and symmetric. Returning to the integral definition of $\delta(\vec{k}_{s_t}-\vec{k}_{i_t})$, it may be shown that $|\delta(\vec{k}_{s_t}-\vec{k}_{i_t})|^2$ should be interpreted as follows;

$$\left|\delta(\vec{k}_{s_t}-\vec{k}_{i_t})\right|^2 = \lim_{A\to\infty} A\,\delta(\vec{k}_{s_t}-\vec{k}_{i_t})$$

Thus, the last term in (14) is recognized to be the magnitude squared of the mean field, i.e.

$$|\langle\vec{E}_s\cdot\hat{e}\rangle|^2 = \lim_{A\to\infty} \frac{A|\Gamma|^2}{(2\pi)^{2(s_\infty-1)}}\,\delta(\vec{k}_{s_t}-\vec{k}_{i_t})\left|\iiint \hat{e}\cdot\vec{j}(k_1,\vec{K}_1)\,\Phi_1(k_{s_z}-k_1,-\vec{K}_2)\,dk_1\,dK_1\right|^2 \tag{15}$$

and so the variance of the scattered field is as follows;

$$\begin{aligned}\mathrm{Var}(\vec{E}_s\cdot\hat{e}) = \lim_{A\to\infty}\frac{A|\Gamma|^2}{(2\pi)^{2s_\infty}}\iiiint &[\vec{j}(k_1,\vec{K}_1)\cdot\hat{e}][\vec{j}^*(-k_2,-\vec{K}_2)\cdot\hat{e}]\\ &\tilde{\bar{\Phi}}_2(k_{s_z}-k_1,-k_{s_z}-k_2,-\vec{K}_1,-\vec{K}_2;\vec{k}_{s_t}-\vec{k}_{i_t})\,dk_1\,dk_2\,d\vec{K}_1\,d\vec{K}_2\end{aligned} \tag{16}$$

where $|\Gamma|^2 = (k_o\eta_o/R_o)^2$, $\bar{\Phi}_2(\Delta\vec{r}_t) = \Phi_2(\Delta\vec{r}_t) - \Phi_2(\infty)$, and

$$\tilde{\bar{\Phi}}_2(\cdot,\cdot;\vec{k}_{s_t}-\vec{k}_{i_t}) = \int\left\{\Phi_2(\cdot,\cdot;\Delta\vec{r}_t)-\Phi_2(\cdot,\cdot;\infty)\right\}\exp[j(\vec{k}_{s_t}-\vec{k}_{i_t})\cdot\Delta\vec{r}_t]\,d\Delta\vec{r}_t \tag{16a}$$

The important point to note about (16) is that the directional characteristics of $\mathrm{Var}(\vec{E}_s\cdot\hat{e})$ are determined essentially by th[illegible]r transform of $\bar{\Phi}_2$ evaluated in the direction $\hat{k}_{s_t}-\hat{k}_{i_t}$. That is, the far field approximation predicts that the variance of the scattered field is proportional to only one Fourier component of $\bar{\Phi}_2$.

The results obtained in this section are not intended to justify the use of the far field approximation when the surface extent and the illumination are both unbounded. Rather, this section is intended to serve two other purposes. First, the technique of treating averages as convolutions in stochastic transform space has been introduced and used for the purpose of familiarization. Second and more important is the fact that the only approximation contained in this section's results is the far field approximation. Thus, it is now possible to obtain similar results from the exact expression for the scattered field and compare them with (15) and (16) without having the comparison tainted by the use of an approximate current. That is, any differences between (15) and (16) and corresponding exact results will be solely a consequence of the far field approximation.

Exact Results for Unbounded Illumination

The exact mean scattered field can be obtained without much difficulty from (1); however, this is not necessary since it can also be derived from $\langle|\vec{E}_s \cdot \hat{e}|^2\rangle$, apart from a sign, by the same approach as demonstrated above. The derivation of a simplified expression for $\langle|\vec{E}_s \cdot \hat{e}|^2\rangle$ from (1) is also not difficult but it is involved and normally it would be better to relegate it to an appendix. However, it does contain a very important simplifying step and so it will be included in the main text. To avoid a minor difficulty [Brown, 1982a], surfaces for which decorrelation implies statistical independence will be assumed.

Using (1) and the technique of converting averages into convolutions in transform space yields the following;

$$<|\vec{E}_s \cdot \hat{e}|^2> = [\eta_o/k_o]^2 \int \cdots \int F\{\ \}_1 \ F\{\ \}_2 \ \Phi_2 \ (-k_1,-k_2,-\vec{K}_1,-\vec{K}_2;\Delta\vec{r}_t)d\vec{r}_{t_1}$$

$$\cdot \ d\vec{r}_{t_2} dk_1 dk_2 d\vec{K}_1 d\vec{K}_2 \ / (2\pi)^{2s_\infty} \tag{17}$$

where $\Delta\vec{r}_t = \vec{r}_{t_1} - \vec{r}_{t_2}$ and

$$F\{\ \}_1 = (2\pi)^{-1} \hat{e} \cdot \nabla_o \times \nabla_o \times \iint \tilde{\vec{J}}(\vec{r}_{t_1}, k_1-\beta_1, \vec{K}_1) G(\vec{R}_{o_t} - \vec{r}_{t_1}, Z_o - \zeta_1) \exp(j\beta_1\zeta_1) d\zeta_1 d\beta_1$$

$$F\{\ \}_2 = (2\pi)^{-1} \hat{e} \cdot \nabla_o \times \nabla_o \times \iint \tilde{\vec{J}}^*(\vec{r}_{t_2}, k_2-\beta_2, \vec{K}_2) G^*(\vec{R}_{o_t} - \vec{r}_{t_2}, Z_o - \zeta_2) \exp(j\beta_2\zeta_2) d\zeta_2 d\beta_2$$

and $\vec{R}_{o_t} = X_o\hat{x} + Y_o\hat{y}$ is the horizontal distance from the origin on the surface to the point of observation. The tilde symbol over $\vec{J}$ and $\vec{J}^*$ denotes the Fourier transform with respect to the stochastic variables. Since both the surface and the illumination are unbounded, (9) may be used along with the following;

$$\tilde{\vec{J}}^*(\vec{r}_{t_2}, k_2-\beta_2, \vec{K}_2) = \left\{ \tilde{\vec{J}}(\vec{r}_{t_2}, -k_2+\beta_2, -\vec{K}_2) \right\}^*$$

Thus, (17) reduces to the following form;

$$<|\vec{E}_s \cdot \hat{e}|^2> = (\eta_o/k_o)^2 \int \cdots \int \left\{ \hat{e} \cdot \nabla_o \times \nabla_o \times \vec{J}(k_1-\beta_1, \vec{K}_1) G(\vec{R}_{o_t} - \vec{r}_{t_1}, Z_o - \zeta_1) \exp[-j\vec{k}_{i_t} \cdot \vec{r}_{t_1} + j\beta_1\zeta_1] \right\}$$

$$\cdot \left\{ \hat{e} \cdot \nabla_o \times \nabla_o \times \vec{J}^*(-k_2+\beta_2, -\vec{K}_2) G^*(\vec{R}_{o_t} - \vec{r}_{t_2}, Z_o - \zeta_2) \exp[+j\vec{k}_{i_t} \cdot \vec{r}_{t_2} + j\beta_2\zeta_2] \right\}$$

$$\Phi_2(-k_1,-k_2,-\vec{K}_1,-\vec{K}_2;\Delta\vec{r}_t) d\vec{r}_{t_1} d\vec{r}_{t_2} d\zeta_1 d\zeta_2 d\beta_1 d\beta_2 dk_1 dk_2 d\vec{K}_1 d\vec{K}_2 \Big/ (2\pi)^{2(s_\infty+1)} \tag{18}$$

The $\vec{r}_{t_1}$-integration involves only the functions G and Φ_2. Using the shift theorem for Fourier transforms and the two-dimensional Fourier transform of G , the $\vec{r}_t$-integration may be expressed as follows;

$$\int G\,\Phi_2 \exp(-j\vec{k}_{i_t}\cdot\vec{r}_{t_1})d\vec{r}_{t_1} = (2\pi)^{-2}\exp(-j\vec{k}_{i_t}\cdot\vec{R}_{o_t})\int \frac{\exp\left[-j\sqrt{k_o^2-(\vec{k}_{i_t}-\vec{k}_t)^2}\;|Z_o-\zeta_1|\right]}{-j\,2\sqrt{k_o^2-(\vec{k}_{i_t}-\vec{k}_t)^2}}$$

$$\cdot\,\tilde{\Phi}_2(\cdot,\cdot;-\vec{k}_t)\exp\left[j\vec{k}_t\cdot(\vec{R}_{o_t}-\vec{r}_{t_2})\right]d\vec{k}_t \qquad (19)$$

where the tilde over Φ_2 denotes the transform from $\Delta\vec{r}_t$ to $\vec{k}_t$ and $(\vec{k}_{i_t}-\vec{k}_t)^2 = (\vec{k}_{i_t}-\vec{k}_t)\cdot(\vec{k}_{i_t}-\vec{k}_t)$. Since the point of observation (X_o,Y_o,Z_o) is to be above any realization of the rough surface, $|Z_o-\zeta_1| = Z_o-\zeta_1$. Substituting these results in (18) and performing the ζ_1-integration gives rise to a delta function, i.e. $\delta\left(\beta_1+\sqrt{k_o^2-(\vec{k}_{i_t}-\vec{k}_t)^2}\right)$, which makes the β_1-integration trivial. Thus, remembering that Z_o must be above the surface, (18) reduces to the following;

$$<|\vec{E}_s\cdot\hat{e}|^2> = (\eta_o/k_o)^2\int\cdots\int\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(k_1+\kappa,\vec{K}_1)\exp[-j\kappa Z_o+j\vec{k}_t\cdot(\vec{R}_{o_t}-\vec{r}_{t_2})]/(-j2\kappa)\right.$$

$$\left.\cdot\exp(-j\vec{k}_{i_t}\cdot\vec{R}_{o_t})\right\}\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}^*(-k_2+\beta_2,-\vec{K}_2)G^*(\vec{R}_{o_t}-\vec{r}_{t_2}.Z_o-\zeta_2)\right.$$

$$\left.\exp[+j\vec{k}_{i_t}\cdot\vec{r}_{t_2}+j\beta_2\zeta_2]\right\}\tilde{\Phi}_2(\cdot,\cdot;-\vec{k}_t)d\zeta_2 d\beta_2 d\vec{r}_{t_2}d\vec{k}_t dk_1 dk_2 d\vec{K}_1 d\vec{K}_2/(2\pi)^{2s_\infty+3} \qquad (20)$$

where $\kappa = [k_o^2-(\vec{k}_{i_t}-\vec{k}_t)^2]^{1/2}$. The $\vec{r}_{t_2}$-integration in (20) yields the two-dimensional Fourier transform of G^* which contains an exponential term whose argument is proportional to $|Z_o-\zeta_2|$. Dropping the magnitude sign since Z_o is to always be greater than ζ_2 and performing the ζ_2-integration in (20) yields a δ-function which simplifies the β_2-integration. The result is as follows;

$$<|\vec{E}_s\cdot\hat{e}|^2> = (\eta_o/k_o)^2 \iiiint\!\!\iint \left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}\,(k_1+\bar{k},\vec{K}_1)\exp\left[-j\bar{k}\,Z_o-j\vec{k}_t\cdot\vec{R}_{o_t}\right]\Big/(-j2\bar{k})\right\}$$

$$\cdot\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(\bar{k}-k_2+\bar{k},-\vec{K}_2)\exp\left[-j\bar{k}\,Z_o-j\vec{k}_t\cdot\vec{R}_{o_t}\right]\Big/(-j2\bar{k})\right\}^*$$

$$\cdot\quad \tilde{\Phi}_2\,(-k_1,-k_2,-\vec{K}_1,-\vec{K}_2;\vec{k}_t-\vec{k}_{i_t})d\vec{k}_t dk_1 dk_2 d\vec{K}_1 d\vec{K}_2\Big/(2\pi)^{2s_\infty+2} \qquad (21)$$

where $\bar{k} = [k_o^2 - \vec{k}_t^{\,2}]^{1/2}$. Regularizing the integrand in (21) so that it does not contain a $\delta(\vec{k}_t - \vec{k}_{i_t})$ dependence entails subtracting and adding $\Phi_2(\cdot,\cdot;\infty)$ <u>prior</u> to transforming from $\Delta\vec{r}_t$ to $\vec{k}_t-\vec{k}_{i_t}$. Thus, with

$$\tilde{\tilde{\Phi}}_2(\cdot,\cdot;\vec{k}_t-\vec{k}_{i_t}) = \int\left[\Phi_2(\cdot,\cdot;\Delta\vec{r}_t) - \Phi_2(\cdot,\cdot;\infty)\right]\exp\left[j(\vec{k}_t-\vec{k}_{i_t})\cdot\Delta\vec{r}_t\right]d\Delta\vec{r}_t$$

and the fact that $\Phi_2(\cdot,\cdot;\infty)$ may be written as a product of marginal characteristic functions, the regularized form of (21) becomes

$$<|\vec{E}_s\cdot\hat{e}|^2> = (\eta_o/k_o)^2\iiiint\!\!\iint\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(k_1+\bar{k},\vec{K}_1)\psi(\vec{k}_t,\vec{R}_o)\right\}\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(-k_2+\bar{k},-\vec{K}_2)\psi(\vec{k}_t,\vec{R}_o)\right\}^*$$

$$\cdot\quad\tilde{\tilde{\Phi}}_2(-k_1,-k_2,-\vec{K}_1,-\vec{K}_2;\vec{k}_t-\vec{k}_{i_t})d\vec{k}_t dk_1 dk_2 d\vec{K}_1 d\vec{K}_2\Big/(2\pi)^{2s_\infty+2}$$

$$+(\eta_o/k_o)^2\iiiint\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(k_1,\vec{K}_1)\psi(\vec{k}_{i_t},\vec{R}_o)\right\}\left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{J}(k_2,\vec{K}_2)\psi(\vec{k}_{i_t},\vec{R}_o)\right\}^*$$

$$\cdot\quad\Phi_1\left(-k_1+\sqrt{k_o^2-\vec{k}_{i_t}^{\,2}},-\vec{K}_1\right)\Phi_1\left(k_2-\sqrt{k_o^2-\vec{k}_{i_t}^{\,2}},\vec{K}_2\right)dk_1 dk_2 d\vec{K}_1 d\vec{K}_2\Big/(2\pi)^{2s_\infty} \qquad (22)$$

where $\vec{R}_o = X_o\hat{x} + Y_o\hat{y} + Z_o\hat{z}$, $\bar{k} = (k_o^2 - \vec{k}_t^{\,2})^{1/2}$, and

$$\psi(\vec{k}_t,\vec{R}_o) = \exp[-j\bar{k}\, Z_o - j\,\vec{k}_t \cdot \vec{R}_{o_t}] \Big/ (-j\, 2\bar{k}) \tag{22a}$$

The last term in (22) is the magnitude squared of the mean scattered field, $|\langle\vec{E}_s \cdot \hat{e}\rangle|^2$, and the following operations demonstrate this fact. Since Φ_1 is real and symmetric in all transform variables, the last term in (22) which results from regularizing (21) is recognized to be the product of one set of integrals and its complex conjugate. Thus, it may be expressed as follows;

$$|\langle\vec{E}_s \cdot \hat{e}\rangle|^2 = \Big|(\eta_o/k_o)\int\!\!\int\Big\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{j}\,(k_1,\vec{K}_1)\psi(\vec{k}_{i_t},\vec{R}_o)\Big\}\Phi_1(-k_1+\sqrt{k_o^2-k_{i_t}^2},-\vec{K}_1)$$

$$\cdot\, dk_1 d\vec{K}_1 \Big/ (2\pi)^{s_\infty}\Big|^2 \tag{23}$$

The function $\psi(k_{i_t},\vec{R}_o)$ is independent of k_1 and $\vec{K}_1$ and is a plane wave. The root of $(k_o^2-\vec{k}_{i_t}^2)^{1/2}$ must be chosen such that $\psi(\vec{k}_{i_t},\vec{R}_o)$ is upward traveling because Z_o is positive; this dictates the following form for $\psi(\vec{k}_{i_t},\vec{R}_o)$;

$$\psi(\vec{k}_{i_t},\vec{R}_o) = \exp\Big[-j|k_{i_z}|\; Z_o - j\vec{k}_{i_t}\cdot\vec{R}_{o_t}\Big]\Big/(-j\,2|k_{i_z}|) \tag{24}$$

where the magnitude signs on k_{i_z} are used because, according to (3), $k_{i_z} \le 0$. The quantities $\nabla_o\times\nabla_o\times\vec{j}\psi$ may be manipulated using vector identities into the form

$$\nabla_o\times\nabla_o\times\vec{j}\,\psi = k_o^2\big[\psi\,\vec{j} + (\vec{j}\cdot\nabla_o)\nabla_o\,\psi\big] \tag{25}$$

By expanding the vector operations in (25) with the aid of (24), it may be shown that

$$\nabla_o\times\nabla_o\times\vec{j}\,\psi = \big[\hat{k}_s^s \times \hat{k}_s^s \times \vec{j}\big]\, k_o\,\psi_s\big/(-j2\cos\theta_i)$$

where θ_i is the angle of incidence measured with respect to the z-axis,

$\hat{k}_s^s$ is a unit vector in the specular direction. i.e.

$$\hat{k}_s^s = (\vec{k}_{i_t} + |k_{i_z}|\hat{z}) / |\vec{k}_{i_t} + |k_{i_z}|\hat{z}| \quad ,$$

and

$$\psi_s = \exp(-j\, k_o R_o \hat{k}_s^s \cdot \hat{k}_s)$$

Substituting these results in (23) and taking $\hat{e}\cdot\hat{k}_s^s = 0$ yields

$$|<\vec{E}_s\cdot\hat{e}>|^2 = \left| \frac{\eta_o \psi_s}{2\cos\theta_i} \iint \hat{e}\cdot\vec{j}(k_1,\vec{K}_1)\,\Phi_1(-k_1+|k_{i_z}|,-\vec{K}_2)\, dk_1 d\vec{K}_1 \Big/ (2\pi)^{s_\infty} \right|^2 \tag{26}$$

and it can be shown that the expression inside the magnitude brackets is identical, except for a minus sign, to the average of $\vec{E}_s\cdot\hat{e}$ obtained from (1). Substracting (26) from (22) gives the following relationship for the variance of the scattered field;

$$\mathrm{Var}(\vec{E}_s\cdot\hat{e}) = (\eta_o/k_o)^2 \iiiint\!\!\int \left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{j}(k_1+\bar{k},\vec{K}_1)\psi(\vec{k}_t,\vec{R}_o)\right\} \left\{\hat{e}\cdot\nabla_o\times\nabla_o\times\vec{j}(-k_2+\bar{k},-\vec{K}_2)\right.$$
$$\left.\cdot\,\psi(\vec{k}_t,\vec{R}_o)\right\}^* \tilde{\tilde{\Phi}}_2(-k_1,-k_2,-\vec{K}_1,-\vec{K}_2;\vec{k}_t-\vec{k}_{i_t})\, d\vec{k}_t\, dk_1 dk_2 d\vec{K}_1 d\vec{K}_2 \Big/ (2\pi)^{2s_\infty+2} \tag{27}$$

Unfortunately, the integrations in (26) and (27) cannot be accomplished in general without knowledge of $\vec{j}$, Φ_1, and Φ_2 .

The important point to remember about (26) and (27) is that they are both exact results. Furthermore, it is not necessary for the point of observation R_o to be infinitely far from the surface. The only restriction on these results is that the point of observation be above any realization of the surface.

Comparison of Exact and Far Field Results

Equations (15) and (16) are results obtained using only the far field approximation while (26) and (27) are exact. A comparison of the two sets will show how the far field approximation breaks down when both the surface and the illumination are unbounded.

The exact result for the mean scattered power, i.e. (26), shows that the mean scattered field is a plane wave propagating in the specular direction, $\hat{k}^s_{s_t} = \hat{k}_{s_i}$ and $\hat{k}^s_{s_z} = -\hat{k}_{i_z}$. Furthermore, it can be shown that (26) yields the correct reflected power when the random surface goes to a plane. The far field approximation for $\langle \vec{E}_s \cdot \hat{e} \rangle$, i.e. (10), yields a spherical wave with an amplitude weighting such that it is nonzero only in the specular direction. In the limit of the random surface approaching a plane, the far field approximation does not reduce to a reflected plane wave traveling in the specular direction. It is interesting to note that although the far field approximation does not produce the plane wave nature of the scattered field, it does contain the proper dependence upon the statistical properties of the surface because in the specular direction $k_{s_z} = |k_{iz}|$.

Equation (26) does not provide any insight into the polarization of the mean scattered field; the polarization is determined entirely by the transformed current $\vec{J}(k_1,\vec{K}_1)$. If there is no significant multiple scattering occurring on the rough surface then $\langle \vec{E}_s \rangle$ will have the same polarization as a field reflected from a perfectly flat surface. In this case $\langle \vec{E}_s \rangle$ will have all of the characteristics of a specularly reflected field except that it will be attenuated by the effects of the surface roughness. If multiple scattering is significant, the polarization of $\langle \vec{E}_s \rangle$ will be different from that of a planar surface reflected field. This result shows that, contrary to previous thoughts [Moore and Williams, 1957], the mean scattered field is not governed by the laws of reflection if multiple scattering on the surface is significant. Furthermore, it is not advisable to refer to the mean scattered field as a specular field since $\langle \vec{E}_s \rangle$ does not always have the characteristics of a true specularly reflected field.

According to (16) the variance of the scattered field is linearly related to the illuminated area and inversely proportional to the second power of the distance from the origin on the surface to the point of observation. More importantly however is the fact that the variance is determined by the Fourier transform of the regularized characteristic function evaluated at $\vec{k}_{s_t} - \vec{k}_{i_t}$, i.e. see (16) and (16a). This is a direct consequence of the fact that if one represents the scattered field in terms of an angular spectrum of plane waves [Clemmow, 1966], the far field approximation leads to a scattered field which is determined by one and only one angle or direction in the spectral representation. In terms of the variance of the scattered field, this means that one obtains a result which is determined by one and only one Fourier component of the regularized characteristic function. Contrast this result with the exact result as given by (27) in which the variance of the scattered field is determined by a weighted integral of <u>all</u> Fourier components of the regularized characteristic function. In fact, the very presence of the $\vec{k}_t$-integration in (27) shows that there is no such thing as a far field for the case of unbounded illumination regardless of how far the point of observation is moved away from the surface. For example, by moving the point of observation away from the surface one merely sees less of the surface waves propagating along the surface. In fact, in the limit of $R_o \to \infty$ the $\vec{k}_t$-integration in (27) is essentially limited to the visible range $(\vec{k}_t^2 < k_o^2)$ because of the $\psi(\vec{k}_t, \vec{R}_o)$ function; but $Var(\vec{E}_s \cdot \hat{e})$ is still determined by a weighted integral of all Fourier components of $\bar{\Phi}_2$ over the visible range of wavenumbers. As an aside, it should be noted that although one does not observe any surface waves as $R_o \to \infty$ they still influence the result through the transformed current $\vec{j}$. As a final point, it can be shown that (27) goes to the proper limits in the cases of a planar surface or a randomly elevated planar surface whereas (16) does not.

It would be very beneficial if the $\vec{k}_t$-integration in (27) could be accomplished without explicit knowledge of either $\vec{j}$ or $\tilde{\tilde{\Phi}}_2$. However, this is not possible primarily because the convergence of the integral is highly dependent upon $\vec{j}$ and $\tilde{\tilde{\Phi}}_2$. This can be seen by noting that if the $\vec{j}$'s and $\tilde{\tilde{\Phi}}_2$ are set to unity, the $\vec{k}_t$-integration is exactly equal to the integral of $|[\hat{e}\cdot\nabla_o\times\nabla_o\times G(R_o)]|^2$ over the x-y plane and this integral does not converge.

It is certainly not surprising that the far field approximation fails when the incident illumination is unbounded. In the case of the mean scattered field, it has been shown that the effects of the surface roughness are correctly predicted by the far field approximation. However, for the variance of the scattered field there is a very fundamental difference between the far field results and the exact results and so the far field approximation is essentially useless.

Bounded Illumination

With the exception of certain fundamental theoretical questions, the case of bounded incident illumination is of most interest. Remembering that the intent here is to obtain general results which do not depend on explicit approximate forms for the surface current, the bounding of the illumination causes a problem because the separation in (9) is no longer valid. In order to understand this difficulty and investigate appropriate approximations, it is necessary to turn to the stochastic Fourier transformed magnetic field integral equation for the modified current $\vec{J}$ [Brown, 1982b]. This equation is obtained by multiplying the magnetic field integral equation by $\exp[jk_1\zeta + j\sum_{n=1}^{\infty}\vec{k}_{n+1}\cdot\nabla^n\zeta]$ and then averaging over all stochastic surface characteristics $\nabla^n\zeta, n=0,1,\cdots$. The averages are converted to convolutions in stochastic transform space and one obtains an infinite dimensional integral equation of the first kind having the following form;

$$(2\pi)^{-s_\infty}\int\cdots\int\tilde{\vec{J}}(\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots)\,\Phi_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots)d\beta_1 d\vec{\beta}_2\cdots$$

$$= \tilde{\vec{J}}_o(k_1,\vec{k}_2)\,\Theta(\vec{r}_t)\exp(-j\vec{k}_{i_t}\cdot\vec{r}_t) + (2\pi)^{-s_\infty-2}\int\cdots\int\Big\{\bar{\bar{\chi}}(\Delta\vec{r}_t,\beta_o,\vec{\beta}_2,\vec{k}_2)$$

$$\Phi_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots;\Delta\vec{r}_t)\Big\}\cdot\tilde{\vec{J}}(\vec{r}_t-\Delta\vec{r}_t,\beta_1,\vec{\beta}_2,\cdots)\;d\Delta\vec{r}_t d\beta_o d\beta_1 d\vec{\beta}_2\cdots \tag{28}$$

The tilde over $\vec{J}$ denotes the stochastic Fourier transform (from $\nabla^n\zeta$ space to $\vec{\beta}_n$ space), $\tilde{\vec{J}}_o$ is the transform of the stochastic part of the physical optics current due to the incident field, i.e. $2\hat{n}\times\vec{H}_i$, and $\bar{\bar{\chi}}$ is a dyadic kernel involving stochastic transforms of Green's function derivatives. When the illumination is unbounded so that $\Theta(\vec{r}_t)=1$ for all $\vec{r}_t$, it is obvious that (9) is valid. However, when the support of the incident illumination is finite, the dependence of $\tilde{\vec{J}}$ on $\vec{r}_t$ is very difficult to obtain. The Fourier transform of (28) with respect to $\vec{r}_t$ yields

$$(2\pi)^{-s_\infty}\int\cdots\int\tilde{\tilde{\vec{J}}}(\vec{k}_t,\beta_1,\vec{\beta}_2,\cdots)\Phi_1(k_1-\beta_1,\vec{k}_2-\vec{\beta}_2,\cdots)d\beta_1 d\vec{\beta}_2\cdots = \tilde{\vec{J}}_o(k_1,\vec{k}_2)\Theta(\vec{k}_t-\vec{k}_{i_t})$$

$$+\,(2\pi)^{-s_\infty-2}\int\cdots\int\left\{\iiint\tilde{\bar{\bar{\chi}}}(\vec{k},\beta_o\vec{\beta}_2,\vec{k}_2)\,\tilde{\Phi}_2(k_1-\beta_o,\beta_o-\beta_1,\vec{k}_2,-\vec{\beta}_2,\cdots;\vec{k}_t-\vec{k})d\vec{k}d\beta_o\right\}$$

$$\cdot\,\tilde{\tilde{\vec{J}}}(\vec{k}_t,\beta_1,\vec{\beta}_2,\cdots)d\beta_1 d\vec{\beta}_2\cdots \tag{29}$$

It is tempting to assume that $\tilde{\tilde{\vec{J}}}$ can be expressed as a product of a k_t-dependent function and one which depends upon $\beta_1,\vec{\beta}_2,\cdots$. However, there is no reason for this form to be valid in general because $\tilde{\bar{\bar{\chi}}}\tilde{\Phi}_2$ cannot similarly be expressed as a product. In fact, it is the presence of the integral term on the rhs of

(29) which greatly complicates matters. This term is important when multiple scattering on the surface is significant; in the absence of multiple scattering $\tilde{\vec{J}}$ has the same $\vec{r}_t$-dependence as the incident illumination. When there is strong multiple scattering present, the $\vec{k}_t$-dependence of $\tilde{\tilde{\vec{J}}}_t$ in (29) is influenced by the statistical properties of the surface through $\tilde{\Phi}_2$ in (29). This result has a simple physical explanation in that multiple scattering implies that the current at any point on the surface is determined by the behavior of the surface in a <u>neighborhood</u> of the point; thus, the support of the current on the surface will differ from the projected area of the incident illumination. One may also visualize the multiple scattering process in terms of rays undergoing at least two reflections on the surface before traveling off into free space; consequently, there will be some rays reflected to the surface outside of the incident illumination support. What this all means is that the $\vec{r}_t$-dependence or the support of the transformed current is determined both by the incidence illumination and the surface roughness when multiple scattering is important.

To overcome this difficulty and yet not reduce the problem to the trivial case of single scattering, it is necessary to assume that the support of the incident illumination projected on the $z=0$ plane is very large compared to λ_o^2. If this is true then $\tilde{\Theta}(\vec{k}_t - \vec{k}_{i_t})$ will be very sharply peaked about $\vec{k}_t = \vec{k}_{i_t}$ and $\tilde{\tilde{\vec{J}}}$ must then necessarily have the same dependence. Physically, this approximation implies that the illuminated area is so large that there is little relative difference between the projected area of the incident wave and the true illuminated area including multiple scattering. Thus, as long as the cross sectional area of the incident beam plane wave projected onto the $z=0$ plane is large compared to λ_o^2, (29) shows that

$$\tilde{\vec{J}}(\vec{r}_t,k_1,\vec{K}) = \vec{j}(k_1,\vec{K})\,\Theta(\vec{r}_t)\exp(-j\vec{k}_{1_t}\cdot\vec{r}_t) \tag{30}$$

is a valid approximation.

Equation (30) is a very important result but one must be careful to understand what it does and does not imply. Equation (30) can be used to compute the statistical moments of the scattered field; it cannot be used to infer the sample-by-sample behavior of the scattered field. Thus, just because $\tilde{\vec{J}}$ has a finite support does not mean that one may use (2), i.e. the far field approximation, rather than (1), i.e. the exact expression for $\vec{E}_s$, because the actual support of the sample-by-sample current may vary significantly from $\Theta(\vec{r}_t)$. This is a point that must be remembered when attempting to make far field measurements of the field scattered by a rough surface. That is, if one bases the far field condition on $\Theta(\vec{r}_t)$, it must be realized that there may well be some samples of $\vec{E}_s$ which do not satisfy this condition. Thus, one would be well advised in a measurement program to exceed any distances based on $\Theta(\vec{r}_t)$.

As it turns out, the statistical moments of the scattered field are the same regardless of whether one uses (1) or (2) in conjunction with (30). However, as noted above there is no rigorous justification for starting with (2). Furthermore, there is a very important point in the development starting with (1) which shows why the far field approximation fails when the illumination is unbounded. To simplify matters, the development will be carried thru for the mean scattered field, however, the principles apply also to all the moments of the scattered field.

Using $\vec{J}(\vec{r})d\vec{r}_t$ for $\vec{J}_s(\vec{r})dS$, the average scattered field from (1) is given by

$$\langle\vec{E}_s(\vec{R}_o)\rangle = -\,j\,\frac{\eta_o}{k_o}\,\nabla_o\times\nabla_o\times\int\langle\vec{J}(\vec{r})G(|\vec{R}_o-\vec{r}|)\rangle\,d\vec{r}_t \tag{31}$$

where

$$<\vec{J}(\vec{r})G(|\vec{R}_o-\vec{r}|)> = \int\int \vec{J}(\vec{r}_t,\zeta,\partial\zeta)G(|\vec{R}_{o_t}-\vec{r}_t|\ ,\ Z_o-\zeta)p_1(\zeta,\partial\zeta)d\zeta d(\partial\zeta) \qquad (32)$$

Converting the integrals in (32) to convolutions in stochastic Fourier transform space yields the following;

$$<\cdot> = (2\pi)^{-(s_\infty+1)}\int\int\int\int \tilde{\vec{J}}(\vec{r}_t,k_1-\beta_o,\vec{K})G(|\vec{R}_{o_t}-\vec{r}_t|,Z_o-\zeta)$$

$$\cdot\ \exp(j\beta_o\zeta)\,\Phi_1(-k_1,-\vec{K})d\zeta d\beta_o dk_1 d\vec{K}$$

Substituting from (30) yields;

$$<\cdot> = (2\pi)^{-(s_\infty+1)}\int\int\int\int \vec{j}(k_1-\beta_o,\vec{K})\ \Phi_1(-k_1,-\vec{K})\ \Theta(\vec{r}_t)G(|\vec{R}_{o_t}-\vec{r}_t|,\ Z_o-\zeta)$$

$$\cdot\ \exp(-\ j\,\vec{k}_{i_t}\cdot\vec{r}_t\ +\ j\beta_o\zeta)d\zeta d\beta_o dk_1 d\vec{K} \qquad (33)$$

Inserting this expression in (31) and rearranging integrations yields the following $\vec{r}_t$-integral;

$$\int\Theta(\vec{r}_t)G(|\vec{R}_{o_t}-\vec{r}_t|\ ,Z_o-\zeta)\exp(-j\vec{k}_{i_t}\cdot\vec{r}_t)d\vec{r}_t$$

which may be rewritten as a convolution of transforms, i.e.

$$(2\pi)^{-2}\int\tilde{\Theta}(-\vec{k}_{i_t}+\vec{k}_t)\tilde{G}(\vec{k}_t\ ,\ Z_o-\zeta)\exp(-j\vec{k}_t\cdot\vec{R}_{o_t})d\vec{k}_t$$

$$= (2\pi)^{-2}\int\tilde{\Theta}(\vec{k}_t-\vec{k}_{i_t})\exp\left[-\,j\vec{k}_t\cdot\vec{R}_{o_t}\ -\ j\sqrt{k_o^2-\vec{k}_t^2}\ |Z_o-\zeta|\right]\left(-j2\sqrt{k_o^2-\vec{k}_t^2}\right)^{-1}d\vec{k}_t \qquad (34)$$

Taking Z_o to always be above the surface implies $|Z_o-\zeta| = Z_o-\zeta$. The resulting dependence of the rhs of (34) on $\vec{R}_o$ corresponds to a generalized plane wave, consequently, the curl curl operation in (31) may be simplified to the following;

$$\nabla_o \times \nabla_o \times (\vec{j}\psi) = -k_o^2 \psi (\hat{k} \times \hat{k} \times \vec{j})$$

where $\psi = \exp\left(-j\vec{k}_t \cdot \vec{R}_{o_t} - j\sqrt{k_o^2 - \vec{k}_t^2}\, Z_o\right) \Big/ \left(-j2\sqrt{k_o^2 - \vec{k}_t^2}\right)$ and

$$\hat{k} = \left(\vec{k}_t + \sqrt{k_o^2 - \vec{k}_t^2}\,\hat{z}\right) \Big/ |\vec{k}_t + \sqrt{k_o^2 - \vec{k}_t^2}\,\hat{z}|$$

Substituting these results in (33) and (31) yields

$$\begin{aligned}<\vec{E}_s(\vec{R}_o)> = &-\frac{k_o \eta_o}{2} \iiiint\!\!\int \left[\hat{k} \times \hat{k} \times \vec{j}(k_1 - \beta_o, \vec{K})\right] \Phi_1(-k_1, -\vec{K})\, \tilde{\Theta}(\vec{k}_t - \vec{k}_{i_t}) \\ &\cdot \exp\left[-j\vec{k}_t \cdot \vec{R}_{o_t} - j\sqrt{k_o^2 - \vec{k}_t^2}\; Z_o + j\left(\beta_o + \sqrt{k_o^2 - \vec{k}_t^2}\right)\zeta\right](k_o^2 - \vec{k}_t^2)^{-1/2} \\ &\cdot d\vec{k}_t\, d\zeta\, d\beta_o\, dk\, d\vec{K} \big/ (2\pi)^{s_\infty + 3} \end{aligned} \tag{35}$$

Since for any realization of the surface $Z_o > \zeta$, the dominant terms in the $\vec{k}_t$-integration are $\tilde{\Theta}$ and the exponential factor containing $\vec{R}_o$. For any $\Theta(\vec{r}_t)$ having finite support it is always possible to take R_o large enough that the $\vec{k}_t$-integral can be asymptotically evaluated via the method of steepest descent [Clemmow, 1966]. However, if $\Theta(\vec{r}_t)$ has infinite support then it dominates the $\vec{k}_t$-integration because $\tilde{\Theta}(\vec{k}_t - \vec{k}_{i_t}) = \delta(\vec{k}_t - \vec{k}_{i_t})$. Since use of the steepest descent method corresponds to the far field approximation [Clemmow, 1966], it is clear that the validity of the far field approximation is entirely dependent upon which of these factors is dominant in the $\vec{k}_t$-integral. It is also interesting to note that the $\vec{k}_t$-integration is essentially of the same form that Hansen and Bailin [1959] studied for the case of a uniformly illuminated deterministic circular aperture. They found that the far field approximation is valid for $R_o \gtrsim D^2/\lambda_o$ where D is the diameter of the aperture. Thus, for $R_o \gtrsim D^2/\lambda_o$ and $\pi(D/2)^2 >> \lambda_o^{\,2}$, the $\vec{k}_t$-integration in (35) may be asymptotically evaluated using the method of steepest descent. Following this, one completes the ζ and β_o integrations in (35) to yield

$$<\vec{E}_s(\vec{R}_o)> = jk_o\eta_o\, G(R_o)\tilde{\Theta}(\vec{k}_{s_t}-\vec{k}_{i_t})\hat{k}_s \times \hat{k}_s \times \int\int \vec{J}(k_1+k_{s_z},\vec{K})\,\Phi_1(-k_1,-\vec{K})dk_1 d\vec{K}/(2\pi)^{s_\infty} \tag{36}$$

where $\hat{k}_s$ is a unit vector in the direction of the point of observation $(\vec{R}_o = R_o\hat{k}_s)$ and $k_{s_z} = k_o\,\hat{z}\cdot\hat{k}_s$ is the product of k_o and the z-component of this unit vector.

This same asymptotic approach may be used to obtain an approximate expression for the variance of the scattered. To account for the limited extent of the incident illumination, it is sufficient to replace $\vec{J}$ by $\vec{J}\,\Theta(\vec{r}_{t_1})$ and $\vec{J}^*$ by $\vec{J}^*\Theta(\vec{r}_{t_2})$ in (18). One then uses the steepest descent method in $\vec{k}$-space to first evaluate the $\vec{r}_{t_1}$-integral and then the $\vec{r}_{t_2}$-integral. Asymptotic evaluation of the $\vec{r}_{t_1}$-integral leads to the following $\vec{k}$-space integration;

$$\int \tilde{\Theta}(\vec{k}_{s_t} - \vec{k}_{i_t} + \vec{k}_2)\tilde{\tilde{\Phi}}_2(-\vec{k}_2)\exp(-j\vec{k}_2\cdot\vec{r}_{t_2})d\vec{k}_2$$

If the illuminated area encompasses many correlation lengths then $\tilde{\Theta}$ is very sharply peaked relative to $\tilde{\tilde{\Phi}}_2$ and the above integral may be approximated as follows;

$$\tilde{\tilde{\Phi}}_2(\vec{k}_{s_t}-\vec{k}_{i_t})\int \tilde{\Theta}(\vec{k}_{s_t}-\vec{k}_{i_t}+\vec{k}_2)\exp(-j\vec{k}_2\cdot\vec{r}_{t_2})d\vec{k}_2 = \tilde{\tilde{\Phi}}_2(\vec{k}_{s_t}-\vec{k}_{i_t})$$

$$\Theta(\vec{r}_{t_2})\exp\left[j(\vec{k}_{s_t}-\vec{k}_{i_t})\cdot\vec{r}_{t_2}\right]$$

Combining the $\vec{r}_{t_2}$-dependent terms from this result with the remaining $\vec{r}_{t_2}$ factors in (18) and expressing the $\vec{r}_{t_2}$-integral as a convolution in $\vec{k}$-space yields the following integral;

$$\int \tilde{\theta}^2 (\vec{k}_{s_t} - \vec{k})\, \tilde{G}(-\vec{k}, Z_o - \zeta) \exp (j\vec{k}\cdot\vec{R}_{o_t}) d\vec{k}$$

When this integral is evaluated by the steepest descent method, one obtains a result proportional to $\tilde{\theta}^2(0)$ which, for uniform illumination, is equal to the illuminated area. The final result for the variance of the scattered field is identical to (16) except that the illuminated area A is finite. Once again, it is the interplay between the Green's function and the illumination support that determines the suitability of the far field approximation.

It should be emphasized that these results are very general in that they do not depend on any special approximation for the current induced on the rough surface. However, these results are only valid if the cross sectional area of the incident illumination beam, as projected onto the $z = 0$ plane, is very large compared to λ_o^2. This simplification minimizes the error due to the difference between the illumination area and the true area on the surface over which the current is nonzero. Furthermore, in the case of the variance of the scattered field, it is also necessary to stipulate that the illuminated area contain very many decorrelation intervals of the surface roughness in order to recover (16). When this simplifying requirement is not met, the problem becomes more difficult and will be left to future studies.

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